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DEPARTMENT OF ECONOMICS

RESEARCH MEMORANDUM

TOWARDS AN AXIOMATIZATION OF ORDERINGS

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35

FEW 552 Ordering

Refereed by Prof.dr. P.H.M. Ruys

TOWARDS AN AXIOMATIZATION OF ORDERINGS

by Ton Storcken[1] [2]
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Februari 1991 version 1

ABSTRACT A set of six axioms for sets of relations is introduced. All well-known sets of specific orderings, such as linear and weak orderings, satisfy these axioms. These axioms impose criteria of closedness with respect to several operations, such as concatenation, substitution and restriction. For operational reasons and in order to link our results with the literature, it is shown that specific generalizations of the transitivity condition give rise to sets of relations which satisfy these axioms. Next we study minimal extensions of a given set of relations which satisfy the axioms. By this study we come to the fundamentals of orderings: They appear to be special arrangements of several types of disorder. Finally we notice that in this framework many new sets of relations have to be regarded as a set of orderings and that it is not evident how to minimize the number of these new sets of orderings.

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§1 PROBLEM, SOLUTION AND ORGANIZATION

In the literature many types of orderings/preferences have been introduced, e.g. linear orderings, weak orderings, semi-orderings, partial orderings, interval orderings, quasi-orderings, tournaments and many less well-known orderings. These orderings have been developed in various fields such as economy, psychology, sociology, operations research, decision theory, discrete mathematics and many others. A lot of research on these orderings was dedicated to model these types of relations and to compare them with each other as well as with graphtheoretical or combinatorial concepts. However, to the best of our knowledge, there has not been an investigation into a system which models all well-known types of orderings. Of course, all orderings have been introduced by imposing criteria on relations in such a way that each type of ordering is determined by its own special set of criteria. Although such a set of criteria can formalize the phenomenon of "linear", "weak", "semi", "partial", "interval" or any other type of ordering one might think of, it does not formalize the phenomenon of "ordering" itself. Up till now, we have no formal criteria which enable us to clarify a set of relations as a set of orderings. Nevertheless, such criteria may exist since we use the word "ordering" in a specific way whenever we deal with relations.

In this paper six criteria are proposed of which we claim that any set of orderings should satisfy them. These criteria are non-triviality of the set of relations and closedness with respect to permutation, conversion, concatenation, restriction, and substitution. A set of relations which satisfies these criteria is said to be classified as a set of orderings. Of course, other criteria might be proposed as well; hence, the expression "towards" in our title. In order to have a good balance between precision and clearness we do not want to be exhaustivily precise.

In finding criteria for the axiom system, we had the following in mind:

(1) the well-known sets of orderings should satisfy the criteria;

(2) between the set of the linear and the set of the weak orderings there should be no other classifiable set of orderings;

(3) the number of classifiable sets of orderings should be minimal (in some sense).

It is, however, inevitable to admit at least denumerably many classified sets of orderings, whenever we stick to these six criteria. We could not find any other suitable criterion or variations of one of these criteria that could decrease the number of classifiable sets of orderings. (see section 6).

On the other hand we feel very pleased not only because of the fact that we have succeeded in a formalization of the phenomenon of ordering, but also because of the fact that this formalization enables us to deal with old problems in a fundamental way. By virtue of this axiom-system Storcken [1989] was able to generalize Arrow's Impossibility Theorem in Social Choice Theory tremendously, bringing the fundamentals of the problem to the front. In Delver, Monsuur & Storcken [1991] the data ordered in a tournament are rearranged to a weak ordering by virtue of the operations mentioned above.

The system proposed here consists of criteria for sets of relations. So sets of relations are classified as sets of orderings. The criteria imposed on linear orderings, weak orderings, etc. are criteria for relations and not for sets of relations. Our motivation for finding criteria for sets of relations is the following. The decision whether a certain relation R is a linear ordering, a weak ordering or any other type of ordering is based on the fact that R belongs to a certain class of relations called, repectively, the set of linear orderings, weak orderings, etc. The decision whether R is an ordering or not depends on the fact whether R belongs to a class of relations which can be conceived as a set of special orderings. Thus we have to determine under what conditions a set of relations should be classifiable as a set of orderings.

In the next section we introduce the primitive notions and formulate the six criteria for sets of relations. Among others we introduce the notion of order-isomorphism. We also state some preliminary results. In section three the notion of transitivity is generalized. It means that special types of paths along a relation can be short-cutted by another special type of paths along that relation. Specific transitivity conditions together with the reflexivity condition determine special classifiable sets of orderings. This result provides us with a sufficient condition by which it is easy to check whether a set of relations is classifiable. Note that many sets of orderings in the literature are described in terms of transitivity conditions. So, by the generalization of the transitivity condition, these sets can be classified as sets of orderings. In section four we discuss minimal extensions. A set W of relations is a minimal extension of a set V, if $V \subset W$, i.e. V is strictly contained in W, V and W are both classifiable as sets of orderings, and there is no classified set U such that $V \subset U \subset W$. We characterize minimal extensions of V as those sets which are closed under all operations on the relations in V and on a relation R not in V.

Given a classifiable set V of orderings and given a relation R not in V a new classifiable set of orderings, being an extension of V, is obtained by applying closure operations on $V \cup \{R\}$. Relations R satisfying some specific properties lead to minimal extensions of V. Intuitively, such an R arranges the elements of its domain disorderly from the standpoint of V. In the extensions however, these arrangements are accepted as order. This leads us to the conclusion that ordering means arranging (according to the operations) several disorderings. Therefore the question whether a relation expresses an ordering depends on the question whether several arrangements, which cannot be analysed further are accepted as ordered on the basis of the operations. For instance, if we accept no disorder at all, then these arrangements are the relations on singletons and only linear orderings or weak orderings are ordering relations. On the other hand, if we accept every possible arrangement, then, for instance, every tournament is an ordering relation. In section five we comment on some new (classified) sets of orderings and on the well-known ones. We give an inclusion diagram. It might be helpfull for the reader to consult this diagram whenever a new type of ordering is classified. In section six we critically discuss the six criteria and several variations on them. Finally we state an open problem.

§2 THE CRITERIA

In set theoretical terms a <u>relation on</u> a given set **A** is defined as a subset of **A x A**. **A** is both the domain and the co-domain of such a relation. Since we introduce operations, such as restriction, concatenation and substitution, by which this domain and co-domain alter, it is necessary to indicate explicitly the domain (and co-domain) together with the relation. Consequently, the empty relation on a set **A** will be different from the empty relation on another set **B**. Moreover, we have to specify all the possible domains (and co-domains).

In this paper we restrict ourselves to finite domains for reasons of simplicity. Along with the development of this theory we shall indicate some problems which are to be solved if e.g. countable domains are admitted. In order to admit for each natural number $n \geq 1$ a domain with cardinality n, we presuppose the existence of an infinite but countable set (<u>universe</u>), say **U**. The set of natural numbers is a possible candidate. The set of all finite subsets of **U** is then the set of all admitted domains:

$$\mathbf{D} := \{A : A \subseteq \mathbf{U} \ \& \ 0 < \# A < \infty\}$$ [3]

From now on we assume that any relation R is a set of ordered pairs (of elements of U) and that in R↓A this R is restricted to those pairs which are in A x A. In fact R↓A will be short for $<\{<x,y> : <x,y> \in R \ \& \ <x,y> \in A \times A\},A>$. Therefore note that $R^1{\downarrow}A = R^2{\downarrow}B$ if and only if $R^1 \cap (A \times A) = R^2 \cap (B \times B)$ and A = B. Let $\mathbf{R} := \{ R{\downarrow}A : R \subseteq \mathbf{U} \times \mathbf{U} \ \& \ A \in \mathbf{D}\}$. Let $\emptyset$ denote the <u>empty set</u>. $Id := \{<x,x> : x \in \mathbf{U}\}$ is the <u>identity relation</u> on **U** and $All := \{<x,y> : x \in \mathbf{U} \ \& \ y \in \mathbf{U}\}$ is the <u>total relation</u> on **U**. For all domains A in **D** $\emptyset{\downarrow}A$, $Id{\downarrow}A$ and $All{\downarrow}A$ denote the <u>empty relation on</u> A, the <u>identity relation on</u> A and the <u>total</u>

[3]Conventionally, we use capital R for relations, other capitals for sets, small letters for elements of **U** or integers, and bold capitals for special sets. Furthermore, #A indicates the cardinality of A.

relation on A respectively. For $x, y \in \mathbf{U}$ and $R{\downarrow}A \in \mathbf{R}$, $\langle x,y \rangle$ is said to be in $R{\downarrow}A$, notation $\langle x,y \rangle \in R{\downarrow}A$, if, $x \in A$, $y \in A$ and $\langle x,y \rangle \in R$. If $\langle x,y \rangle \in R{\downarrow}A$, then we say that x is as least as good as y with respect to $R{\downarrow}A$. Furthermore,

$$R{\downarrow}A = \{\langle x,y \rangle : \langle x,y \rangle \in R \ \& \ \langle x,y \rangle \in A \times A\}{\downarrow}A$$
$$= (R \cap (A \times A)){\downarrow}A.$$

DEFINITION 2.1 OPERATIONS

Let A, B and C be (three) domains in **D**, such that A and B are disjoint. Let $R{\downarrow}A$, $R'{\downarrow}A$ and $R''{\downarrow}B$ be relations in **R**. Let σ be a permutation of **U**. Let $x \in A$.

2.1.1 $R{\downarrow}A \cup R'{\downarrow}A := (R \cup R'){\downarrow}A$ is the union of $R{\downarrow}A$ and $R'{\downarrow}A$.

2.1.2 $R{\downarrow}A \cap R'{\downarrow}A := (R \cap R'){\downarrow}A$ is the intersection of $R{\downarrow}A$ and $R'{\downarrow}A$.

2.1.3 $R{\downarrow}A \circ R'{\downarrow}A := \{\langle x,y \rangle$: There is a $z \in A$ such that $\langle x,z \rangle \in R{\downarrow}A$ and $\langle z,y \rangle \in R'{\downarrow}A\}{\downarrow}A$ is the composition of $R{\downarrow}A$ with $R'{\downarrow}A$.

2.1.4 For all $k \geq 1$: $[R{\downarrow}A]^{k+1} := (R{\downarrow}A) \circ [R{\downarrow}A]^k$. $[R{\downarrow}A]^0 := Id{\downarrow}A$.

2.1.5 $\bar{c}R{\downarrow}A := ((U \times U)-R){\downarrow}A$ is the complement relation of $R{\downarrow}A$.

2.1.6 $\bar{v}R{\downarrow}A := \{\langle x,y \rangle : \langle y,x \rangle \in R\}{\downarrow}A$ is the converse relation of $R{\downarrow}A$.

2.1.7 $\bar{s}R{\downarrow}A := R{\downarrow}A \cap (\bar{v}R{\downarrow}A)$ is the symmetric part of $R{\downarrow}A$.

2.1.8 $\bar{a}R{\downarrow}A := R{\downarrow}A \cap (\bar{c}(\bar{v}R{\downarrow}A))$ is the asymmetric part of $R{\downarrow}A$.

2.1.9 $\bar{n}R{\downarrow}A := R{\downarrow}A \cap (\bar{c}Id{\downarrow}A)$ is the non-diagonal part of $R{\downarrow}A$.

2.1.10 $\bar{r}R{\downarrow}A := R{\downarrow}A \cup Id{\downarrow}A$ is the reflexive closure of $R{\downarrow}A$.

2.1.11 $\bar{t}R{\downarrow}A := \cup\{[R{\downarrow}A]^k : k \in \{0,1,2,..\}\}$ is the transitive closure of $R{\downarrow}A$.

2.1.12 $\bar{o}R{\downarrow}A := \emptyset{\downarrow}A$; $\bar{o}$ is called the constant empty operation on **R**.

2.1.13 $\sigma R{\downarrow}A := \{\langle \sigma(x),\sigma(y) \rangle : \langle x,y \rangle \in R{\downarrow}A\}{\downarrow}\sigma(A)$ is the permutation under σ of $R{\downarrow}A$.

2.1.14 $(R{\downarrow}A)|C := (R \cap (A \times A)) \downarrow C$ is the domain alternation to C of $R{\downarrow}A$.

If $C \subseteq A$, then $(R{\downarrow}A)|C = R{\downarrow}C$ and is called the restriction of $R{\downarrow}A$ to C.

If $A \subseteq C$, then $(R{\downarrow}A)|C$ is called the extension of $R{\downarrow}A$ to C.

2.1.15 $R{\downarrow}A \gg R''{\downarrow}B$:=

$((R{\downarrow}A)\,|\,(A \cup B)) \cup ((R''{\downarrow}B)\,|\,(A \cup B)) \cup ((A \times B){\downarrow}(A \cup B))$ is the <u>concatenation of</u> $R{\downarrow}A$ with $R''{\downarrow}B$.

2.1.16 $Sub(R{\downarrow}A,x,R''{\downarrow}B) := (R \cap ((A-\{x\}) \times (A-\{x\})) \cup (R'' \cap (B \times B) \cup (B \times \{a \in A - \{x\} : \langle x,a\rangle \in R\}) \cup (\{a \in A - \{x\} : \langle a,x\rangle \in R\})) \downarrow (A \cup B) - \{x\}$ is the <u>substitution of</u> $R''{\downarrow}B$ <u>on</u> x <u>in</u> $R{\downarrow}A$.

■

In order to reduce the use of parentheses we introduce priorities between the introduced operations. We assign highest priority to the monadic operations, $\bar{c}$, $\bar{v}$, $\bar{s}$, $\bar{a}$, $\bar{n}$, $\bar{r}$, $\bar{t}$, $\bar{o}$ and σ. Next, decreasing priority is assigned to domain alteration, concatenation, substitution, composition, intersection and union, in the order listed. For example, $\bar{c}R{\downarrow}A \cup \bar{v}R{\downarrow}A = (\bar{c}(R{\downarrow}A)) \cup (\bar{v}(R{\downarrow}A))$, which is different from $\bar{c}((R{\downarrow}A) \cup \bar{v}(R{\downarrow}A))$.

The monadic operations $\bar{c}$, $\bar{v}$, $\bar{s}$, $\bar{a}$, $\bar{r}$, $\bar{t}$ and σ are well-known from the literature, see e.g. Rouben & Vincke [1985]. The conversion of R is often indicated by $\check{R}$; sometimes it is called the inversion of R. In Storcken [1989] it is shown that there are 64 monadic operations on relations. They play an important rôle in the development of our generalized transitivity condition, to be introduced later on. However, for our purpose we can restrict ourselves to the nine monadic operations defined above.

Union, intersection and composition are well-known binary Set Theoretical operations. Domain alteration is just an extension of restriction which is also a well-known operation in Set Theory. Concatenation is a binary operation which has already been introduced in the literature under various other names. Rosenstein [1982] calls it summation and Jónsson [1982] calls it lexicographic product. We call it concatenation, because the result of the operation is a linear chain of two relations.

Substitution means what you would expect: In $Sub(R{\downarrow}A,x,R''{\downarrow}B)$ $R''{\downarrow}B$ plays the rôle of x in $R{\downarrow}A$, where $R''{\downarrow}B$ and $(R{\downarrow}A)|A-\{x\}$ remain unchanged. $R''{\downarrow}B$ is the substitute for x.

Let V be a subset of R. We define criteria for a set V to be classified as a set of orderings.

Firstly, we demand that for each possible domain, the set of relations in V on that domain is not trivial. Secondly, we demand that the names of elements in $\mathbf{U}$ are not important, in other words that V is closed under every permutation of $\mathbf{U}$. Thirdly, we impose that converting all pairs of an ordering in V results in a relation that is again in V, i.e. that V is closed under conversion. In the fourth place we requir that restrictions of in ordering in V are again in V, in other words, that V is closed under restriction. In the fifth place, we impose that having two disjoint orderings in V the concatenation of these two is again in V. Note that the concatenation strictly prefers every element occurring in the first ordering above every element occurring in the second one. Finally, we demand that every non-discriminating ordering in V can be substituted for an element occurring in an ordering of V, yielding an ordering that is again in V. In other words, V is closed under substitution of non-discriminating orderings in V. $R{\downarrow}A \in V$ is <u>non-discriminating</u> if for all permutations σ of AxA $\sigma(R{\downarrow}A) = R{\downarrow}A$. Note that $R{\downarrow}A$ is non-discriminating if and only if $R{\downarrow}A \in \{All{\downarrow}A, \varnothing{\downarrow}A\}$. We denote the <u>set of all non-discriminating relations of</u> V by $\mathbf{N}(V)$. Hence, if $R{\downarrow}\{x\} \in V$, then $R{\downarrow}\{x\} \in \mathbf{N}(V)$.

DEFINITION 2.2 <u>CLASSIFYING CRITERIA</u>

Let $V \subseteq \mathbf{R}$. V is <u>a classified (or classifiable as a) set of orderings</u>, if V satisfies (2.2.1) up to (2.2.6).

<u>CRITERION 2.2.1</u> V is <u>non-trivial</u>, i.e. for all $A \in \mathbf{D}$ there are $R' \subseteq \mathbf{U}x\mathbf{U}$ and $R \subseteq \mathbf{U}x\mathbf{U}$ such that $R'{\downarrow}A \in V$ and $R{\downarrow}A \notin \mathbf{R} - V$.

<u>CRITERION 2.2.2</u> V is <u>closed under permutation</u>, i.e. for all permutations σ of $\mathbf{U}$ and for all $R{\downarrow}A \in V$: $\sigma R{\downarrow}A \in V$.

CRITERION 2.2.3 V is closed under conversion, i.e. for all $R{\downarrow}A \in V$: $\bar{V}R{\downarrow}A \in V$.

CRITERION 2.2.4 V is closed under restriction, i.e. for all $R{\downarrow}A \in V$ and for all $B \subseteq A$ with $B \neq \emptyset$: $(R{\downarrow}A)|B \in V$.

CRITERION 2.2.5 V is closed under concatenation, i.e. for all $R{\downarrow}A$ and all $R'{\downarrow}B$ in V with $A \cap B = \emptyset$: $R{\downarrow}A \gg R'{\downarrow}B \in V$.

CRITERION 2.2.6 V is closed under substitution of non-discriminating relations in V, i.e. for all $R{\downarrow}A \in V$, for all $x \in A$ and for all $R'{\downarrow}B \in N(V)$ with $A \cap B = \emptyset$: $Sub(R{\downarrow}A,x,R'{\downarrow}B) \in V$.

■

We think that the criteria above are intuitively appealing although perhaps the first five are more plausible than the last one. Nevertheless, let us explain the intuitive idea's in formulating the criteria. When we think of orderings, we expect that the names of the elements which are to be ordered have no effect on the ordering. This expectation is modelled by the closedness under permutations. When thinking of orderings we expect to arrange the elements in some sense from "good" to "bad" or from "bad" to "good"; from "left" to "right" or from "right" to "left". Since "good" and "bad" and "left" and "right" are arbitrary qualifications, it is appealing that whenever we accept an arrangement from "good" to "bad" as an ordering, we should also accept a similar arrangement from "bad" to "good" as an ordering. This is modelled by the closedness under conversion. We think of an ordering as an arrangement whose parts are arrangements which are ordered as well. This idea is modelled by the closedness under restrictions. We also expect that given any two orderings one can make a new one by prefering every element in the first one strictly above any element in the second one. This idea is expressed by the closedness under concatenation. When we think of orderings, then we think that for each possible domain $A \in D$ there are relations on A which we accept as an ordering and there are relations on A which we do not accept as an ordering on A. This idea is modelled by the non-triviality criterion.

The closedness under substitution expresses that the number of elements in a non-discriminating part of an ordering, whose elements are ordered equivalently (in that ordering), is not significant. The set of weak orderings whose indifference classes are equal or smaller than e.g. 10 elements, satisfies criterion 1 to 5. It is a set which is strictly between the set of linear orderings and the set of weak orderings. So if we drop criterion 6 our set of criteria no longer sarisfies condition (2) as mentioned in §1. For variations of this closedness under substitution we refer to §6.
Before we investigate which sets of relations are classifiable, we first prove some preliminary results.

THEOREM 2.3 Let $V \subseteq \mathbf{R}$.

Then (2.3.1) and (2.3.2) are equivalent, where

2.3.1 V is a classified set of orderings, and

2.3.2 V is closed under conversion, restriction, concatenation and substitution, and furthermore,
either for all $x \in \mathbf{U}$: $Id{\downarrow}\{x\} \in V$,
or for all $x \in \mathbf{U}$: $\emptyset{\downarrow}\{x\} \in V$.

PROOF (2.3.1) ⇒ (2.3.2). Assume 2.3.1. By the non-triviality of V with respect to singleton domains and the closedness under permutations the "either, or"-part of (2.3.2) follows evidently.
(2.3.2) ⇒ (2.3.1) Assume 2.3.2. The "either, or"-part of (2.3.2) and the closedness under restriction imply that either all $R{\downarrow}A \in V$ are <u>reflexive</u>, i.e. $\bar{r}R{\downarrow}A = R{\downarrow}A$ or all $R{\downarrow}A \in V$ are <u>irreflexive</u>, i.e. $\bar{n}R{\downarrow}A = R{\downarrow}A$. So for all $A \in \mathbf{D}$, V does not contain all relations on A. Let $A = \{a_1,a_2,\ldots,a_n\}$. Then by the closedness under concatenation

$Id{\downarrow}\{a_1\} \gg Id{\downarrow}\{a_2\} \gg \ldots \gg Id{\downarrow}\{a_n\} \in V$ or

$\emptyset{\downarrow}\{a_1\} \gg \emptyset{\downarrow}\{a_2\} \gg \ldots \gg \emptyset{\downarrow}\{a_n\} \in V$. So V is non-trivial.

Let $A \in \mathbf{D}$. Let $x \in A$. Let $y \in \mathbf{U}-A$. Let $\sigma_{xy} : \mathbf{U} \to \mathbf{U}$ be such that $\sigma_{xy}(x) = y$, $\sigma_{xy}(y) = x$ and $\sigma_{xy}(z) = z$ for all $z \in \mathbf{U} - \{x,y\}$. Let $R{\downarrow}A \in V$. In order to prove the closedness of V under permutation it is sufficient to prove $\sigma_{xy}R{\downarrow}A \in V$. Now since all relations in V are reflexive or all are irreflexive we have $\sigma_{xy}R{\downarrow}A = Sub(R{\downarrow}A,x,Id{\downarrow}\{y\})$ or $\sigma_{xy}R{\downarrow}A = Sub(R{\downarrow}A,y,\emptyset{\downarrow}\{y\})$. Hence we are done by the closedness under substitution and the "either, or"-part.

■

Note that the proof of theorem 2.3 implies

COROLLARY 2.4 Let $V \subseteq \mathbf{R}$ be a classified set of orderings. Then either all relations in V are reflexive or all relations in V are irreflexive.

■

Algebraically, it is evident how to introduce the notion of ordermorphism and order-isomorphism.

DEFINITION 2.5 MORPHISMS

Let $V,W \subseteq \mathbf{R}$ be two classified sets of orderings. Let $h : V \to W$. Then h is said to be an ordermorphism if,

2.5.1 h preserves domains, i.e. for all $R{\downarrow}A \in V$ and all $R'{\downarrow}B \in W$: If $h(R{\downarrow}A) = R'{\downarrow}B$, then $A = B$,

2.5.2 h commutates with conversion, i.e. for all $R{\downarrow}A \in V$: $h(\bar{v}R{\downarrow}A) = \bar{v}h(R{\downarrow}A)$,

2.5.3 h commutates with restriction, i.e. for all $R{\downarrow}A \in V$ and all $B \subseteq A$, with $B \neq \emptyset$: $h((R{\downarrow}A)|B) = h(R{\downarrow}A)|B$,

2.5.4 h preserves concatenation, i.e. for all $R{\downarrow}A \in V$, $R{\downarrow}B \in V$, with $A \cap B = \emptyset$: $h(R{\downarrow}A \gg R{\downarrow}B) = h(R{\downarrow}A) \gg h(R{\downarrow}B)$, and

2.5.5 h preserves substitution, i.e. for all $R{\downarrow}A \in V$, all $x \in A$ and all $R'{\downarrow}B \in \mathbf{N}(V)$, with $A \cap B = \emptyset$: $h(Sub(R{\downarrow}A,x,R'{\downarrow}B)) = Sub(h(R{\downarrow}A),x,h(R'{\downarrow}B))$.

Furthermore, h is said to be an order-isomorphism if h is a bijective ordermorphism.

■

In the definition of ordermorphism we did not demand explicitly the commutation with permutation. The following theorem explains why.

THEOREM 2.6 Let V and W be classified sets of orderings. Let $h : V \to W$ be an ordermorphism. Then h <u>commutates with permutation</u>, i.e. for all permutations σ on $\mathbf{U}$ and all $R{\downarrow}A \in V$: $h(\sigma R{\downarrow}A) = \sigma h(R{\downarrow}A)$. In addition, h(V) can be classified as a set of orderings.

PROOF Let $h : V \to W$ be an ordermorphism. Let $A \in \mathbf{D}$. Let $x \in A$. Let $y \in \mathbf{U} - A$. Let $\sigma_{xy} : \mathbf{U} \to \mathbf{U}$ such that $\sigma_{xy}(x) = y$, $\sigma_{xy}(y) = x$ and $\sigma_{xy}(z) = z$ for $z \in \mathbf{U} - \{x,y\}$. Let $(R{\downarrow}A) \in V$. In order to prove the commutation of h with permutations it suffices to prove that $h(\sigma_{xy}R{\downarrow}A) = \sigma_{xy}h(R{\downarrow}A)$.
Note that for all classified sets X of orderings and for all two relations $R'{\downarrow}A \in X$ and $R''{\downarrow}\{y\} \in X$: $\sigma_{xy}R'{\downarrow}A = Sub(R'{\downarrow}A,x,R''{\downarrow}\{y\})$. So, since h preserves domains and substitution, it follows that h commutates with permutations.
h(V) is evidently a classifiable set of orderings.

■

Although one might expect that there are many ordermorphisms, the following results show that this is not the case. So, the conditions to be satisfied by any ordermorphism are very restrictive.

LEMMA 2.7 Let V and W be two classified sets of orderings. Let $h : V \to W$ be an ordermorphism. Let $A,X \in \mathbf{D}$. Let $a,b \in A$ and $x,y \in X$. Let $R{\downarrow}A$, $R'{\downarrow}X \in V$ such that

(1) $\langle a,b\rangle \in R{\downarrow}A$ iff $\langle x,y\rangle \in R'{\downarrow}X$,

(2) $\langle b,a\rangle \in R{\downarrow}A$ iff $\langle y,x\rangle \in R'{\downarrow}X$, and

(3) $a = b$ iff $x = y$.

Then $\langle a,b\rangle \in h(R{\downarrow}A)$ iff $\langle x,y\rangle \in h(R'{\downarrow}X)$.

PROOF Let σ be a permutation of $\mathbf{U}$, such that $\sigma(a) = x$, $\sigma(b) = y$, $\sigma(x) = a$ and $\sigma(y) = b$. Then $\sigma((R{\downarrow}A)|\{a,b\}) = (R'{\downarrow}X)|\{x,y\}$.
Hence, $\sigma(h(R{\downarrow}A)|\{a,b\}) = h(R'{\downarrow}X)|\{x,y\}$.

So, $\langle x,y\rangle \in h(R'{\downarrow}X)$ iff $\langle \sigma(x),\sigma(y)\rangle \in h(R{\downarrow}A)|\{a,b\}$
iff $\langle a,b\rangle \in h(R{\downarrow}A)$.

■

LEMMA 2.8 Let V and W be two classified sets of orderings. Let $h : V \to W$ be an ordermorphism. Let $R{\downarrow}A \in V$. Then

2.8.1 $\bar{a}h(R{\downarrow}A) = \bar{a}R{\downarrow}A$ and $\overline{av}h(R{\downarrow}A) = \overline{av}R{\downarrow}A$, and

2.8.2 $\overline{ns}h(R{\downarrow}A) \in \{ \varnothing{\downarrow}A, \overline{ns}R{\downarrow}A, \overline{nsc}R{\downarrow}A, \overline{nsca}R{\downarrow}A\}$.

PROOF Suppose $\langle x,y\rangle \in \bar{a}R{\downarrow}A$. Hence
$(R{\downarrow}A)|\{x,y\} = (R{\downarrow}A)|\{x\} \gg (R{\downarrow}A)|\{y\}$.

$$\begin{aligned} \text{Then } h(R{\downarrow}A)|\{x,y\} &= h((R{\downarrow}A)|\{x,y\}) \\ &= h((R{\downarrow}A)|\{x\} \gg (R{\downarrow}A)|\{y\}) \\ &= h((R{\downarrow}A)|\{x\}) \gg h((R{\downarrow}A)|\{y\}) \\ &= h(R{\downarrow}A)|\{x\} \gg h(R{\downarrow}A)|\{y\}. \end{aligned}$$

So, $\langle x,y\rangle \in \bar{a}h(R{\downarrow}A)$. Therefore, $\bar{a}R{\downarrow}A \subseteq \bar{a}h(R{\downarrow}A)$. Similarly,

$\overline{av}R{\downarrow}A \subseteq \overline{av}h(R{\downarrow}A)$.

By lemma 2.7 it follows that $h(\overline{nsca}R{\downarrow}A) = \overline{nsca}h(R{\downarrow}A)$. Since h preserves domains it follows that $\bar{a}R{\downarrow}A = \bar{a}h(R{\downarrow}A)$ and

$\overline{av}R{\downarrow}A = \overline{av}h(R{\downarrow}A)$. In order to prove 2.8.2, by lemma 2.7 we can distinguish four cases.

<u>Case 1</u> $\overline{ns}R{\downarrow}A \subseteq h(R{\downarrow}A)$ and $\overline{nsc}R{\downarrow}A \subseteq h(R{\downarrow}A)$.

In that case it follows by 2.8.1 that $\overline{ns}h(R{\downarrow}A) = \overline{nsca}R{\downarrow}A$ and $\overline{nsc}h(R{\downarrow}A) = \varnothing{\downarrow}A$.

<u>Case 2</u> $\overline{ns}R{\downarrow}A \subseteq h(R{\downarrow}A)$ and $\overline{nsc}R{\downarrow}A \cap h(R{\downarrow}A) = \varnothing$.

In that case it follows by 2.8.1 that $\overline{ns}h(R{\downarrow}A) = \overline{ns}R{\downarrow}A$ and $\overline{nsc}h(R{\downarrow}A) = \overline{nsc}R{\downarrow}A$.

<u>Case 3</u> $\overline{ns}R{\downarrow}A \cap h(R{\downarrow}A) = \varnothing$ and $\overline{nsc}R{\downarrow}A \subseteq h(R{\downarrow}A)$.

In that case it follows by 2.8.1 that $\overline{ns}h(R{\downarrow}A) = \overline{nsc}R{\downarrow}A$ and $\overline{nsc}h(R{\downarrow}A) = \overline{ns}R{\downarrow}A$.

Case 4 $\overline{ns}R{\downarrow}A \cap h(R{\downarrow}A) = \emptyset$ and $\overline{nsc}R{\downarrow}A \cap h(R{\downarrow}A) = \emptyset$.

In that case it follows by 2.8.1 that $\overline{ns}h(R{\downarrow}A) = \emptyset{\downarrow}A$ and

$\overline{nsc}R{\downarrow}A = \overline{nsca}(R{\downarrow}A)$. ■

An immediate result of lemma 2.8 and 2.7 is:

COROLLARY 2.9 Let V and W be classified sets of orderings and $h : V \to W$ an ordermorphism.

Then $h \in \{\bar{n},\bar{r},\overline{ncv},\overline{rcv},\overline{ncav},\overline{cav},\bar{a},\overline{ra}\}$.

If additionally h is bijective, then $h \in \{\bar{n},\bar{r},\overline{ncv},\overline{rcv}\}$. ■

Now let $\bar{i}$, defined by $\bar{i}R{\downarrow}A := R{\downarrow}A$ for all $R{\downarrow}A \in \mathbf{R}$, be the <u>identity operator</u>. The following theorem shows that the set of all possible ordermorphisms is even smaller.

THEOREM 2.10 Let V and W be classified sets of orderings. Let $h : V \to W$.

2.10.1 h is an ordermorphism iff $h \in \{\bar{i},\bar{a},\overline{cv},\overline{cav}\}$.

2.10.2 h is an order-isomorphism iff $h \in \{\bar{i},\overline{cv}\}$.

PROOF 2.10.1 implies 2.10.2, since in the case that $\bar{a}$ or $\overline{cav}$ are bijective, they are equal to $\overline{cv}$ or $\bar{i}$. Furthermore, $\bar{i}$ and $\overline{cv}$ are bijective. (2.10.1) The "if"-part is straight forward and therefore omitted.
"Only-if"-part. Suppose h is an order-morphism. Then by corollary 2.9 $h \in \{\bar{n},\bar{r},\overline{ncv},\overline{rcv},\overline{ncav},\overline{cav},\bar{a},\overline{ra}\}$. There are four cases.

Case 1 $\mathbf{N}(V) = \{All{\downarrow}\{x\} : x \in \mathbf{U}\}$. Hence
$\mathbf{N}(h(V)) \in \{\{All{\downarrow}\{x\} : x \in \mathbf{U}\}, \{\emptyset{\downarrow}\{x\} : x \in \mathbf{U}\}\}$ and
$\bar{a}h(R{\downarrow}A) = \bar{a}R{\downarrow}A = \bar{n}R{\downarrow}A$ for all $R{\downarrow}A \in V$. Then

$\bar{n} = \overline{ncv} = \overline{ncav} = \bar{a} = \overline{cv}$ and $\bar{r} = \overline{rcv} = \overline{cav} = \overline{ra} = \bar{I}$. So

$h \in \{\bar{I}, \overline{cv}\}$.

<u>Case 2</u> $\mathbf{N}(V) = \{\emptyset\downarrow\{x\} : x \in \mathbf{U}\}$; similar to case 1.

<u>Case 3</u> $\mathbf{N}(V) = \{All\downarrow\{X\} : X \in \mathbf{D}\}$.

Hence $\mathbf{N}(h(V)) \in \{\{All\downarrow X : X \in \mathbf{D}\}, \{\emptyset\downarrow X : X \in \mathbf{D}\}\}$. Then

$h \notin \{r\overline{cv}, \overline{ncav}, \bar{n}, \overline{ra}\}$, $\bar{r} = \bar{I}$ and $\overline{ncv} = \overline{cv}$. So

$h \in \{\bar{I}, \bar{a}, \overline{cv}, \overline{cav}\}$.

<u>Case 4</u> $\mathbf{N}(V) = \{\emptyset\downarrow X : X \in \mathbf{D}\}$; similar to case 3.

■

The proof of the following theorem is elementary and therefore left to the reader.

THEOREM 2.11 Let I be a collection of indices. Let V_i for all $i \in I$ be classified as a set of orderings and let $V := \cap\ \{V_i : i \in I\}$ be non-empty. Then V can be classified as a set of orderings.

■

§3 TRANSITIVITY

We are familiar with the preference types "strict preference", "indifference" and "incomparability". Many transitivity properties are defined in terms of these notions. For instance, quasi-transitivity: if x is strictly preferred to y and y is strictly preferred to z, then x is strictly preferred to z. Or, for instance, semi-transitivity: if x is strictly preferred to y, y strictly preferred to z and z and w are indifferent, then x is strictly preferred to w. Or interval-transitivity: if x is strictly preferred to y, y and z are indifferent, and z is strictly preferred to w, then x is strictly preferred to w. (See Roubens & Vincke [1985] or Blair & Pollack [1979]).

Intuitively, our notion of transitivity is a generalization of the foregoing notions and boiles down to the following: a path along a relation whose individual steps correspond to a specific sequence of preference types, can be short-cutted by a path whose individual steps correspond to a subsequence of another specific sequence of preference types. Below we will define what we mean by a preference type, by a path along a relation whose individual steps correspond to a (sub)sequence of preference types and the notion of transitivity.

A preference type corresponds with a specific part of the relation and is described by monadic operations, e.g. $\bar{a}$, $\bar{s}$

or $\overline{cav}$ (the composition of $\bar{v}$, $\bar{a}$ and $\bar{c}$). $\bar{a}$ constructs the asymmetric part of a relation. Hence $\bar{a}R{\downarrow}A$ indicates strict preference. $\bar{s}$ constructs the symmetric part of a relation;

hence $\bar{s}R{\downarrow}A$ indicates indifference. $\overline{cav}$ is the complement of the conversed asymmetric part. We take the following sets of monadic operators or preference types into account:

$\mathrm{Pr} := \{\bar{a},\overline{ns},\overline{nsc},\overline{ra},\overline{rs},\overline{rsc},\bar{n},\overline{ncv},\overline{nsca},\bar{r},\overline{rcv},\overline{ncav},\overline{sca},\overline{cav}\}$,

$\tilde{\mathrm{Pr}} := \{\bar{a},\overline{ra},\overline{rs},\overline{nsc},\overline{rsc},\bar{r},\overline{ncv},\overline{rcv},\overline{sca},\overline{cav}\}$ and

$\bar{P}r := \{\bar{a}, \overline{rs}, \overline{ns}, \overline{nsc}, \bar{r}, \bar{n}, \overline{ncv}, \overline{nsca}, \overline{ncav}, \overline{sca}, \overline{cav}\}$. For a motivation of these choices the reader is referred to Storcken [1989].

DEFINITION 3.1 Let $R{\downarrow}A \in \mathbf{R}$. Let $C,B \in \mathbf{D}$ such that $\{x_0,x_1,\ldots,x_k\} = C \subseteq B \subseteq A$. Let $f_1,f_2,\ldots,f_k \in \mathbf{Pr}$

3.1.1 A <u>word</u> over the <u>alphabet</u> $\mathbf{Pr}$ is the <u>concatenation</u> of zero or more symbols of $\mathbf{Pr}$.
If $w = f_1f_2\ldots f_k$ then $\bar{v}(w) = f_k\ldots f_2f_1$ is the <u>reversal</u> of w.
$\mathbf{Pr}^+ := \{w : w$ is a word over $\mathbf{Pr}$ of positive length$\}$.
(Whenever $f \in \mathbf{Pr}$ is the composition of say $\bar{g}_1,\bar{g}_2,\ldots,\bar{g}_t$, this composition is denoted by a joint bar over the symbols in f: $f := \overline{g_1g_2\ldots g_k}$).

3.1.2 Suppose $w,w' \in \mathbf{Pr}^+$, $w = f_1f_2\ldots f_k$ and $w' = g_1g_2\ldots g_m$. w is <u>embedded</u> in w', if there is a function $h : \{1,\ldots,k\} \to \{1,2,\ldots,m\}$ such that $h(i) < h(j)$ for all $1 \le i < j \le k$ and $f_i = g_{h(i)}$ for all $1 \le i \le k$.

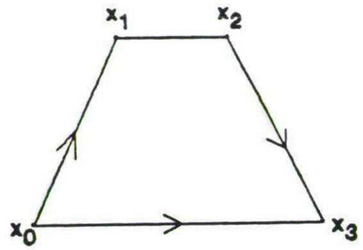
3.1.3 $\pi = \langle x_0,x_1,\ldots,x_k\rangle$ is a <u>path</u> (<u>from</u> x_0 <u>to</u> x_k) (<u>along</u> $R{\downarrow}A$) (<u>in</u> B) (<u>of type</u> $w = f_1f_2\ldots f_k$) if $\langle x_t,x_{t+1}\rangle \in f_{t+1}R{\downarrow}A$ for all $0 \le t \le k-1$. Path π is a <u>cycle</u> if $x_0 = x_k$.
$\bar{v}(\pi) := \langle x_k,x_{k-1},\ldots,x_2,x_1,x_0\rangle$ is the <u>reversal of</u> π.

3.1.4 Let $\pi = \langle x_0,x_1,\ldots,x_k\rangle$ be a path along $R{\downarrow}A$ of type w, $\pi' = \langle y_0,y_1,\ldots,y_n\rangle$ a path along $R{\downarrow}A$ and $w' \in \mathbf{Pr}^+$. π' is a w'-<u>short cut</u> of π if π' is of type w', $\{y_0,y_1,\ldots,y_n\} \subseteq C$, $y_0 = x_0$, $x_k = y_n$ and w' is embedded in w.

■

EXAMPLE 3.2 Let $A \in \mathbf{D}$, $x_0,x_1,x_2,x_3 \in A$, $\langle x_0,x_1\rangle \in \bar{a}R{\downarrow}A$, $\langle x_1,x_2\rangle \in \bar{s}R{\downarrow}A$, $\langle x_2,x_3\rangle \in \bar{a}R{\downarrow}A$ and $\langle x_0,x_3\rangle \in \bar{a}R{\downarrow}A$. Then $\langle x_0,x_1,x_2,x_3\rangle$ is a path from x_0 to x_3 along $R{\downarrow}A$ of type $\bar{a}\bar{s}\bar{a}$. And $\langle x_0,x_3\rangle$ is a $\bar{a}$-short cut of $\langle x_0,x_1,x_2,x_3\rangle$ along $R{\downarrow}A$.

DIAGRAM



■

Intuitively, a word w is embedded in w′, whenever we can find a subsequence of symbols in w′, with the same ordering of symbol occurrences as in w′, which is equal to w.

Let us make two remarks about definition 3.1.

(1) Let $\pi = \langle x_0, x_1, \ldots, x_k \rangle$ be a path along $R{\downarrow}A$ of type $w = f_1 f_2 \ldots f_k$, then w is a transition sequence of preference types occurring in π. So $\langle x_{i-1}, x_i \rangle$ has preference type f_i, i.e., $\langle x_{i-1}, x_i \rangle \in f_i R{\downarrow}A$ for all $i \in \{1,2,..,k\}$.

(2) Of course a type of a path is not unique. For instance,

let $w_1 = \bar{i}\bar{a}\bar{r}\overline{cav}$, $w_2 = \overline{cav}^4$ and $w_3 = \bar{i}^2\bar{r}\overline{cav}$[1] and and π a path along $R{\downarrow}A$. If π is of type w_1, then it is

also of type w_2 and w_3 because $\bar{i}R{\downarrow}A \subseteq \overline{cav}R{\downarrow}A$,

$\bar{a}R{\downarrow}A \subseteq \overline{cav}R{\downarrow}A$ and $\bar{r}R{\downarrow}A \subseteq \overline{cav}R{\downarrow}A$.

The following lemma is used later on and shows some relationship between paths and their types. The proof is elementary and therefore left to the reader.

LEMMA 3.3 Let $\pi = \langle x_0, x_1, \ldots, x_k \rangle$ be a path along $R{\downarrow}A \in \mathbf{R}$ of type $w \in \mathbf{Pr}^+$. Let $w_1, w_2 \in \mathbf{Pr}^+$ and let σ be a permutation of $\mathbf{U}$.

3.3.1 $\bar{v}(\pi)$ is of type $\bar{v}(w)$.

3.3.2 If w_1 is embedded in w_2, then $\bar{v}(w_1)$ is embedded in $\bar{v}(w_2)$.

---

[1] $\overline{cav}^4 := \overline{cav}\ \overline{cav}\ \overline{cav}\ \overline{cav}$, $\bar{i}^2 := \bar{i}\bar{i}$

3.3.3 $\sigma(\pi)$:= $<\sigma(x_0),\sigma(x_1),\ldots,\sigma(x_k)>$ is a path along $\sigma R{\downarrow}A$ of type w.

■

Next we define transitivity which together with reflexivity suffices to yield classifiable sets of orderings (See theorem 3.5). Although a more general transitivity condition is defined in Storcken [1989] we define here a restricted version in order to obtain classifiable sets of orderings. The need of these restrictions is pointed out in Storcken [1989].

DEFINITION 3.4 TRANSITIVITY

Let $R{\downarrow}A \in \mathbf{R}$. Let $w = f_1f_2\ldots f_k \in Pr^+$ and $w' = g_1g_2\ldots g_n \in Pr^+$. $R{\downarrow}A$ is said to be $<w,w'>$-transitive, if:

3.4.1 For every path π along $R{\downarrow}A$ of type w there is a w"-short cut along $R{\downarrow}A$ such that w" is embedded in w',

3.4.2 For every path π along $R{\downarrow}A$ of type $\bar{v}(w)$ there is a w"-short cut along $R{\downarrow}A$ such that w" is embedded in $\bar{v}(w')$,

3.4.3 $f_1,f_2,\ldots,f_k \in \bar{P}r$ and $g_1,g_2,\ldots,g_n \in \bar{P}r$, and

3.4.4 If there is an $i \in \{1,2,\ldots,k\}$ such that

$f_i \in \{\bar{a},\overline{ra},\bar{r},\overline{ncv},\overline{rcv},\overline{cav}\}$, then there is a

$j \in \{1,2,\ldots,n\}$ such that $g_j \in \{\bar{a},\bar{r},\bar{n},\overline{ncv},\overline{ncav},\overline{cav}\}$.

■

(3.4.1) is the essential transitivity requirement. (3.4.2), (3.4.3) and (3.4.4) are necessary to yield classifiable sets of orderings. (3.4.2) is necessary to guarantee that sets of relations satisfying the transitivity condition are closed under conversion. (3.4.3) is necessary for the closure under substitution and (3.4.4) for the closure under concatenation. The need of (3.4.2), (3.4.3) and (3.4.4) will appear in theorem 3.5.

THEOREM 3.5 Let $w, w' \in \mathbf{Pr}^+$.

Let $V := \{R{\downarrow}A \in \mathbf{R} : R{\downarrow}A$ is reflexive and $\langle w,w'\rangle$-transitive$\}$ be nonempty. Then V can be classified as a set of orderings.

PROOF (3.5.1) For all $x \in \mathbf{U}$: $Id{\downarrow}\{x\} \in V$.

Suppose π is a path along $Id{\downarrow}\{x\}$ of type w. We have to prove that there is a w"-short cut π' where w" is embedded in w'. Since V is non-empty there are $A \in \mathbf{D}$, $y \in A$ and $R{\downarrow}A \in V$. Consider $\sigma : \mathbf{U} \to \mathbf{U}$ such that $\sigma(x) = y$, $\sigma(y) = x$ and $\sigma(z) = z$ for all $z \in \mathbf{U} - \{x,y\}$. Then $\sigma(\pi)$ is a path along $Id{\downarrow}\{y\}$ of type w, by lemma 3.3.3. Since $R{\downarrow}A$ is reflexive it follows that $\sigma(\pi)$ is a path along $R{\downarrow}A$ of type w. Hence, by the assumptions on V there is a w"-short cut of $\sigma(\pi)$, say π', such that w" is embedded in w'. But then π' is a w"-short cut of $\sigma(\pi)$ along $Id{\downarrow}\{y\}$. Hence, by lemma 3.3.3 $\sigma(\pi')$ is a w"-short cut of π along $Id{\downarrow}\{x\}$ such that w" is embedded in w'.

(3.5.2) V is closed under conversion.

Let $R{\downarrow}A \in V$. It is sufficient to prove that $\bar{v}R{\downarrow}A \in V$. Let π be a path along $\bar{v}R{\downarrow}A$ of type w. It is sufficient to prove that there is a w"-short cut, say π', of π along $\bar{v}R{\downarrow}A$, such that w" is embedded in w'. By lemma 3.3 $\bar{v}(\pi)$ is a path along $R{\downarrow}A$ of type $\bar{v}(w)$.

Since (3.4.2) holds for $R{\downarrow}A$ there is a w"-short cut, say π', of $\bar{v}(\pi)$ such that w" is embedded in $\bar{v}(w')$. So, by lemma 3.3.2 $\bar{v}(\pi')$ is a path along $\bar{v}R{\downarrow}A$ of type $\bar{v}(w")$ which is embedded in $\bar{v}\bar{v}(w')$. Clearly, $\bar{v}(\pi')$ is a $\bar{v}(w")$-short cut of π along $\bar{v}R{\downarrow}A$ such that $\bar{v}(w")$ is embedded in w'.

(3.5.3) V is closed under restriction.

This follows immediately from the definition of transitivity.

(3.5.4) V is closed under concatenation.

This follows immediately from the definition of transitivity, in particular (3.4.4).

(3.5.5) V is closed under substitution.

Suppose $R{\downarrow}A \in V$, $R'{\downarrow}B \in \mathbf{N}(V)$, $a \in A$ and $A \cap B = \emptyset$. Let $R"{\downarrow}C := Sub(R{\downarrow}A,a,R'{\downarrow}B)$. Let $\pi = \langle c_0,c_1,\ldots,c_k\rangle$ be a path along $R"{\downarrow}C$ of type w. It is sufficient to prove that there is a w"-short cut of π, say π', along $R"{\downarrow}C$ with w" embedded in

w'. Let $w = f_1 f_2 \ldots f_k$ and $w' = g_1 g_2 \ldots g_n$.

Consider $\tau : \{c_0, c_1, \ldots, c_k\} \rightarrow A$, defined by

$$\tau(c_i) = c_i \text{ if } c_i \in A - \{a\}, \text{ and}$$
$$\tau(c_i) = a \text{ if } c_i \in B.$$

Then since $f_1, f_2, \ldots, f_k \in \tilde{P}r$, $\tau(\pi) := <\tau(c_0), \ldots, \tau(c_k)>$ is a path along $R{\downarrow}A$ of type w. It can therefore be cutted short by a path $\pi' = <a_0, a_1, \ldots, a_l>$ of type w'' along $R{\downarrow}A$, where w'' is embedded in w'.

Consider $\sigma : \{a_0, a_1, \ldots, a_l\} \rightarrow C$, defined by

$$\sigma(a_0) = c_0,$$
$$\sigma(a_l) = c_k, \text{ and}$$
$$\text{for all } 0 < i < l, \text{ if } a_i \in A - \{a\}, \text{ then } \sigma(a_i) = a_i, \text{ else } \sigma(a_i) = b;$$

where $b \in B \cap \{c_0, c_1, \ldots, c_k\}$ if $B \cap \{c_0, c_1, \ldots, c_k\} \neq \emptyset$, else $b \in B$.

Then $\sigma(\pi') = <\sigma(a_0), \ldots, \sigma(a_l)>$ is a path of type w'' along $R''{\downarrow}C$ because of (3.4.3). $\sigma(\pi')$ is a w''-short cut of π such that w'' is embedded in w'.

■

Let $R{\downarrow}A \in \mathbf{R}$. $R{\downarrow}A$ is said to be <u>strongly complete</u> if for all $x,y \in A$: $<x,y> \in R{\downarrow}A$ or $<y,x> \in R{\downarrow}A$. $R{\downarrow}A$ is said to be <u>antisymmetric</u> if for all $x,y \in A$ with $x \neq y$: either $<x,y> \notin R{\downarrow}A$ or $<y,x> \notin R{\downarrow}A$. It is straightforward to prove that $\mathbf{C} := \{R{\downarrow}A \in \mathbf{R} : R{\downarrow}A \text{ is strongly complete}\}$ can be (3.6) classified as a set of orderings. It follows from 2.6 and 2.10 that $\overline{cv}\mathbf{C} := \{\overline{cv}R{\downarrow}A : R{\downarrow}A \in \mathbf{C}\}$ can be classified as sets of orderings. Since $\mathbf{C}$ is closed under conversion it follows that $\overline{cv}\mathbf{C} = \bar{c}\mathbf{C}$. Note that by definition a relation $R{\downarrow}A$ in $\mathbf{R}$ is strongly complete if and only if $\bar{c}R{\downarrow}A$ is irreflexive and So $\overline{cv}\mathbf{C} = \bar{c}\mathbf{C}$ is the set of all irreflexive and antisymmetric relations, hence the set of all asymmetric relations, is classified as a set of orderings. Furthermore, we have that $\overline{rc}\mathbf{C}$ is the set of reflexive and antisymmetric relations. It is straightforward to prove that it can be classified as a set of orderings. As a consequence of 2.11 it now follows that

$\mathbf{T} := \mathbf{C} \cap \overline{rc\mathbf{C}}$ is classified as a set of orderings. Note (3.7)
that $\mathbf{T} = \{R{\downarrow}A \in \mathbf{R} : R{\downarrow}A$ is antisymmetric and strongly complete$\}$. $\mathbf{T}$ is often called the set of <u>tournaments</u>.

EXAMPLE 3.8 Let $R{\downarrow}A \in \mathbf{R}$ be reflexive.

$R{\downarrow}A$ is <u>transitive</u> if $R{\downarrow}A \circ R{\downarrow}A \subseteq R{\downarrow}A$.

So, $R{\downarrow}A$ is transitive iff $\bar{I}R{\downarrow}A \circ \bar{I}R{\downarrow}A \subseteq \bar{I}R{\downarrow}A$

iff $R{\downarrow}A$ is $\langle \bar{I}^2, \bar{I}\rangle$-transitive.

From theorem 2.11 and 3.5 it follows that the following sets are classifiable as a set of ordering.

$\mathbf{P} := \{R{\downarrow}A \in \mathbf{R} : R{\downarrow}A$ is reflexive and transitive$\}$, the set of <u>partial orderings</u>. (3.8.1)

$\mathbf{W} := \{R{\downarrow}A \in \mathbf{R} : R{\downarrow}A$ is strongly complete and transitive$\}$, the set of <u>weak orderings</u>. (3.8.2)

$\mathbf{L} := \{R{\downarrow}A \in \mathbf{R} : R{\downarrow}A$ is strongly complete, antisymmetric and transitive$\}$, the set of <u>linear orderings</u>. (3.8.3)

$R{\downarrow}A$ is quasi transitive if $\bar{a}R{\downarrow}A \circ \bar{a}R{\downarrow}A \subseteq \bar{a}R{\downarrow}A$.

So, $R{\downarrow}A$ is quasi transitive iff $R{\downarrow}A$ is $\langle \bar{a}^2, \bar{a}\rangle$-transitive.

From theorem 2.11 and 3.5 it follows that the set of <u>quasi orderings</u> $\mathbf{Q} := \{R{\downarrow}A \in \mathbf{R} : R{\downarrow}A$ is strongly complete and quasi transitive$\}$ can be classified as a set of orderings. (3.8.4)

$R{\downarrow}A$ is <u>acyclic</u> if for all $t \geq 1$: $[\bar{a}R{\downarrow}A]^t \subseteq \overline{cav}R{\downarrow}A$.

So, $R{\downarrow}A$ is acyclic iff for all $t \geq 1$: $R{\downarrow}A$ is $\langle \bar{a}^t, \overline{cav}\rangle$-transitive. Hence, it follows from 2.11 and 3.5 that the set of reflexive and <u>acyclic relations</u> $\mathbf{A} := \{R{\downarrow}A \in \mathbf{R} : R{\downarrow}A$ is reflexive and acyclic$\}$ can be classified as a set of orderings. (3.8.5)

Note that $\mathbf{L} \subset \mathbf{W} \subset \mathbf{P} \subset \mathbf{A}$ and $\mathbf{L} \subset \mathbf{W} \subset \mathbf{Q} \subset \mathbf{A}$, where "$\subset$" indicates strict inclusion. Furthermore, $\mathbf{P} \not\subseteq \mathbf{Q}$ and $\mathbf{Q} \not\subseteq \mathbf{P}$.

■

We have just seen how theorem 3.5 can be used in order to show that a set of relations, satisfying a transitivity condition, is classifiable as a set of orderings. In section 5 other less familiar transitivity conditions will be discussed. The following example is another illustration of how theorem 3.5 can be applied.

EXAMPLE 3.9 Let $V := \{R{\downarrow}A \in C : \bar{c}R{\downarrow}A \circ \bar{c}R{\downarrow}A \subseteq \bar{c}R{\downarrow}A\}$.

V is the set of strongly complete and negatively transitive relations. Note that for $R{\downarrow}A$ in C:

$\bar{c}R{\downarrow}A \circ \bar{c}R{\downarrow}A \subseteq \bar{c}R{\downarrow}A$ iff $\overline{ncv}R{\downarrow}A \circ \overline{ncv}R{\downarrow}A \subseteq \overline{ncv}R{\downarrow}A$.

Hence, $V := \{R{\downarrow}A \in C : R{\downarrow}A$ is $\langle\overline{ncv}^2,\overline{ncv}\rangle$-transitive$\}$ is classifiable as a set of orderings by theorem 3.5.

Let $T_k := \{R{\downarrow}A \in T :$ if $\pi = \langle x_0,x_1,\ldots,x_l\rangle$ is a path along $R{\downarrow}A$ such that $x_l = x_0$, then $\#\{x_0,x_1,\ldots,x_l\} \le k\}$.

For $R{\downarrow}A \in T$ and $k \in \{3,4\}$:

$R{\downarrow}A \in T_k$ iff $R{\downarrow}A$ is $\langle\bar{a}^{k-1},\overline{cav}\rangle$-transitive.

So T_3 and T_4 are classifiable sets of orderings. (3.9.1)

Later on we will show that T_k is classifiable for all $k \ge 3$. ■

§4 MINIMAL EXTENSIONS

In the previous section we introduced a transitivity condition by which many well-known sets of orderings can be classified. So, the criteria proposed in section 2 are at least weak enough to admit these well-known sets of orderings as a model. In order to develop some intuitive ideas about the proposed criteria we have to deduce some logical consequences of these criteria. Here we will focus an so called minimal extensions.

DEFINITION 4.1 <u>MINIMAL EXTENSION</u>

Let V,W be two sets of relations which both can be classified as sets of orderings.
W is an <u>extension of</u> V, if $V \subset W$, i.e. $V \subseteq W$ and $V \neq W$.
W is a <u>minimal extension of</u> V, notation $V \underset{m}{\subsetneq} W$, if W is an extension of V and for all extensions W′ of V with $W' \subseteq W$: $W = W'$.

■

So, if W is a minimal extension of V, then there is no classifiable set of orderings W′ such that $V \subset W' \subset W$.

EXAMPLE 4.2 $\mathbf{L} \underset{m}{\subsetneq} \mathbf{W}$.

Suppose $\mathbf{L} \subset W \subseteq \mathbf{W}$ and W can be classified as a set of orderings. Then there is a relation $R{\downarrow}A \in W \subseteq \mathbf{W}$ which is not in $\mathbf{L}$. Hence, $R{\downarrow}A$ is transitive, strongly complete and not antisymmetric. So, there are $x,y \in A$ with $x \neq y$, such that $All{\downarrow}\{x,y\} = (R{\downarrow}A)|\{x,y\}$. So, $All{\downarrow}\{x,y\} \in W$ since W is closed under restriction. Now by the closedness under substitution it follows that for all $B \in \mathbf{D}$: $All{\downarrow}B \in W$.
Notice that every weak ordering in $\mathbf{W}$ is the concatenation of total indifference relations: If $R'{\downarrow}C \in \mathbf{W}$, then
$R'{\downarrow}C = All{\downarrow}X_1 \gg All{\downarrow}X_2 \gg \ldots \gg All{\downarrow}X_k$ for some partition $X_1, X_2, \ldots, X_k$ of C. Hence, $\mathbf{W} \subseteq W$ by the closedness under concatenation.

■

Example 4.2 shows that minimal extensions do exist. In order to construct minimal extensions we need closure operators with respect to permutation, conversion, restriction, concatenation and substitution. The application of these closure operators in a specific order on a given set V of (ir)reflexive relations yields $\Phi(V)$, the smallest classifiable set of orderings containing V. From this result (corollary 4.6) it follows that if V is a classifiable set of orderings and $R{\downarrow}A \in \mathbf{R} - V$ satisfies a certain condition, then $\Phi(V \cup \{R{\downarrow}A\})$ is a minimal extension of V. Furthermore, we will prove the converse: If $V \subsetneq_m W$, then there is a relation $R{\downarrow}A$ such that $W = \Phi(V \cup \{R{\downarrow}A\})$. After all, we may conclude that the minimal extensions can be characterized by elementary algebraic techniques.

DEFINITION 4.3 CLOSURE OPERATORS

Let V be a set of (ir)reflexive relations.

$\Sigma_{perm}(V) := \{\tau R{\downarrow}A : \tau$ is a permutation of $\mathbf{U}$ and $R{\downarrow}A \in V\}$. $\Sigma_{perm}(V)$ is called the closure under permutation of V.

$\Sigma_{conv}(V) := V \cup \bar{v}(V)$. $\Sigma_{conv}(V)$ is called the closure under conversion of V.

$\Sigma_{rest}(V) := \{(R{\downarrow}A)|B : \emptyset \neq B \subseteq A$ and $R{\downarrow}A \in V\}$. $\Sigma_{rest}(V)$ is called the closure under restriction of V.

$\Sigma_{conc}(V) := \{R^1{\downarrow}A^1 \gg R^2{\downarrow}A^2 \gg \ldots \gg R^k{\downarrow}A^k : k \in \{1,2,\ldots\}$, for all $i \in \{1,2,\ldots,k\}$, $R^i{\downarrow}A^i \in V$ and $A^i \cap A^j = \emptyset$ for all $1 \le i < j \le k\}$. $\Sigma_{conc}(V)$ is called the closure under concatenation of V.

$\Omega(V)$ is the closure under substitution of $\mathbf{N}(V)$; it is defined as follows:

$$\Omega(V) = \begin{cases} \{\emptyset{\downarrow}\{x\} : x \in \mathbf{U}\} \text{ if } \mathbf{N}(V) \subseteq \{\emptyset{\downarrow}\{x\} : x \in \mathbf{U}\}, \\ \{Id{\downarrow}\{x\} : x \in \mathbf{U}\} \text{ if } \mathbf{N}(V) \subseteq \{Id{\downarrow}\{x\} : x \in \mathbf{U}\}, \\ \{All{\downarrow}X : X \in \mathbf{D}\} \text{ if there is a } X \in \mathbf{D}, \text{ with} \\ \qquad \#X \ge 2 \text{ and } All{\downarrow}X \in \mathbf{N}(V), \\ \{\emptyset{\downarrow}X : X \in \mathbf{D}\} \text{ if there is a } X \in \mathbf{D}, \text{ with} \\ \qquad \#X \ge 2 \text{ and } \emptyset{\downarrow}X \in \mathbf{N}(V). \end{cases}$$

$\Sigma_{subs}(V) := \{R{\downarrow}A :$ There is a partition $A^1, A^2, \ldots, A^k$ of A such that for every $i \in \{1,2,\ldots,k\}$ there is a relation $R^i{\downarrow}A^i \in \Omega(V)$ and a relation $R'{\downarrow}B \in V$ with $B = \{b_1, b_2 \ldots, b_n\}$, such that for all $<x,y> \in R{\downarrow}A$ with $x \in A^i$ and $y \in A^j$ either $i = j$ and $<x,y> \in R^i{\downarrow}A^i$ or $i \neq j$ and $<b_i,b_j> \in R'{\downarrow}B\}$.

$\Sigma_{subs}(V)$ is called the <u>closure under substitution</u> of V. ■

It is straightforward to prove the following lemma which verifies that the closure operators of definition 4.3 have the intended properties.

LEMMA 4.4 Let V,W be sets of (ir)reflexive relations such that $V \subseteq W$. Then

4.4.1 $\Sigma_{perm}(V)$ is closed under permutation and if W is closed under permutation, then $\Sigma_{perm}(V) \subseteq W$;

4.4.2 $\Sigma_{conv}(V)$ is closed under conversion and if W is closed under conversion, then $\Sigma_{conv}(V) \subseteq W$;

4.4.3 $\Sigma_{rest}(V)$ is closed under restriction and if W is closed under restriction, then $\Sigma_{rest}(V) \subseteq W$;

4.4.4 $\Sigma_{conc}(V)$ is closed under concatenation and if W is closed under concatenation, then $\Sigma_{conc}(V) \subseteq W$;

4.4.5 $\Omega(V)$ is closed under substitution;

4.4.6 $\Sigma_{subs}(V)$ is closed under substitution and if W is closed under substitution and restriction, then $\Sigma_{subs}(V) \subseteq W$. ■

The following lemma shows that the composition of several operators is more or less commutative.

LEMMA 4.5 Let $V \subseteq \mathbf{R}$ be a set of (ir)reflexive relations.

4.5.1 $\Sigma_{conv}\Sigma_w(V) = \Sigma_w\Sigma_{conv}(V)$ for $w \in \{perm,rest,conc,subs\}$;

4.5.2 $\Sigma_{perm}\Sigma_{rest}(V) \subseteq \Sigma_{rest}\Sigma_{perm}(V)$;

4.5.3 $\Sigma_{perm}\Sigma_{conc}(V) \subseteq \Sigma_{conc}\Sigma_{perm}(V)$;

4.5.4 $\Sigma_{rest}\Sigma_{conc}(V) \subseteq \Sigma_{conc}\Sigma_{rest}(V)$;

4.5.5 $\Sigma_{perm}\Sigma_{subs}(V) \subseteq \Sigma_{subs}\Sigma_{perm}(V)$;

4.5.6 $\Sigma_{rest}\Sigma_{subs}(V) \subseteq \Sigma_{subs}\Sigma_{rest}(V)$;

4.5.7 $\Sigma_{subs}\Sigma_{conc}(V) \subseteq \Sigma_{conc}\Sigma_{subs}(V)$.

PROOF Since $\bar{v}$ is a monadic operator which commutates with every permutation, restriction, concatenation and substitution (4.5.1) follows evidently.

For all $R{\downarrow}A \in \mathbf{R}$, for all $B \subseteq A$, $B \neq \emptyset$ and for all permutations τ of U: $\tau((R{\downarrow}A)|B) = \tau(R{\downarrow}A)|\tau(B)$. So, (4.5.2) follows.

For all $R{\downarrow}A$, $R'{\downarrow}B \in \mathbf{R}$ with $A \cap B = \emptyset$ and for all permutations τ of U: $\tau(R{\downarrow}A \gg R'{\downarrow}B) = \tau R{\downarrow}A \gg \tau R'{\downarrow}B$. So, (4.5.3) follows.

For all $R{\downarrow}A$, $R'{\downarrow}B \in \mathbf{R}$ with $A \cap B = \emptyset$ and for all $C \subseteq A \cup B$, $C \neq \emptyset$:

$$(R{\downarrow}A \gg R'{\downarrow}B)|C = \begin{cases} (R{\downarrow}A)|C \text{ if } C \subseteq A \\ (R'{\downarrow}B)|C \text{ if } C \subseteq B \\ (R{\downarrow}A)|C \cap A \gg (R'{\downarrow}B)|C \cap B \text{ otherwise.} \end{cases}$$

So, (4.5.4) follows.

For all $R{\downarrow}A$, $R'{\downarrow}B \in \mathbf{R}$ with $A \cap B = \emptyset$, for all $a \in A$ and for all permutations τ of U:

$\tau\mathrm{Sub}(R{\downarrow}A,a,R'{\downarrow}B) = \mathrm{Sub}(\tau R{\downarrow}A,\tau(a),\tau R'{\downarrow}B)$. So, (4.5.5) follows.

For all $R{\downarrow}A$, $R'{\downarrow}B \in \mathbf{R}$ with $A \cap B = \emptyset$, for all $a \in A$ and for all $C \subseteq A \cup B - \{a\}$ with $C \neq \emptyset$:

$$\mathrm{Sub}(R{\downarrow}A,a,R'{\downarrow}B)|C = \begin{cases} (R{\downarrow}A)|C \text{ if } C \subseteq A \\ (R'{\downarrow}B)|C \text{ if } C \subseteq B \\ \mathrm{Sub}((R{\downarrow}A)|((C \cap A) \cup \{a\}),a,(R'{\downarrow}B)|(B \cap C)) \\ \qquad \text{otherwise.} \end{cases}$$

So, (4.5.6) follows.

For all $R{\downarrow}A$, $R'{\downarrow}B$ and $R''{\downarrow}C$ in $\mathbf{R}$ with $A \cap B = \emptyset$ and $C \cap (A \cup B) = \emptyset$ and for all $a \in A \cup B$:

$$\mathrm{Sub}(R{\downarrow}A \gg R'{\downarrow}B,a,R''{\downarrow}C) = \begin{cases} \mathrm{Sub}(R{\downarrow}A,a,R''{\downarrow}C) \gg R'{\downarrow}B \text{ if } a \in A \\ R{\downarrow}A \gg \mathrm{Sub}(R'{\downarrow}B,a,R''{\downarrow}C) \text{ if } a \in B \end{cases}$$

So, (4.5.7) follows.

■

COROLLARY 4.6 Suppose V is a set of (ir)reflexive relations, W is a classified set of orderings and $V \subseteq W$. Let $\Phi(V) := \Sigma_{conc}\Sigma_{subs}\Sigma_{rest}\Sigma_{conv}(V)$. Then $\Phi(V) \subseteq W$ and $\Phi(V)$ can be classified as a set of orderings.

PROOF By lemma 4.4 it follows that $\Phi(V) \subseteq W$. From lemma 4.4 it also follows that for all $V' \subseteq R$ such that V' is a non-empty set of (ir)reflexive relations, and for all $w \in \{rest, conv, subs, conc\}$: $\Sigma_w \Sigma_w(V') = \Sigma_w(V')$. Hence, it follows from lemma 4.5 that $\Sigma_w(\Phi(V)) \subseteq \Phi(V)$ for all $w \in \{rest, conv, subs, conc\}$. Since the converse holds by definition, it follows that $\Phi(V) = \Sigma_w(\Phi(V))$ for each such w. Therefore, it follows from lemma 4.4 and theorem 2.3 that $\Phi(V)$ can be classified as a set of orderings.

■

THEOREM 4.7 Let V,W be classified as sets of orderings. $V \subset_m W$ if and only if there is a relation $R{\downarrow}B \in W - V$ with $\#B \geq 2$ and for all $C \subset B$, $C \neq \emptyset$: $(R{\downarrow}B)|C \in V$, such that $W = \Phi(V \cup \{R{\downarrow}B\}) = \Sigma_{conc}\Sigma_{subs}(V \cup \{\bar{v}R{\downarrow}B,\ R{\downarrow}B\})$.

PROOF ("If"-part) Let $W = \Phi(V \cup \{R{\downarrow}B\})$, where $R{\downarrow}B$ satisfies the conditions mentioned above. Then $V \subseteq W$ and W can be classified as a set of orderings. Let $V \subset W' \subseteq W$ such that W' can be classified as a set of orderings. Let $R'{\downarrow}D \in W' - V$. Because D is finite we may suppose that $(R'{\downarrow}D)|C \in V$ for all $C \subset D$, $C \neq \emptyset$. Since $W' \subseteq \Phi(V \cup \{R{\downarrow}B\})$, it follows that $R'{\downarrow}D \in \Sigma_{perm}(V \cup \{R{\downarrow}B, \bar{v}R{\downarrow}B\}) - V$. Hence, $R{\downarrow}B \in W'$ and $W' = W$.
("only if"-part) Let $V \subset_m W$. From the finiteness of all possible domains the existence of $R{\downarrow}B$ follows evidently. From corollary 4.6 it then follows that $\Phi(V \cup \{R{\downarrow}B\}) \subseteq W$. By the minimality of W we have $\Phi(V \cup \{R{\downarrow}B\}) = W$. Since $\Sigma_{rest}\Sigma_{conv}(V \cup \{R{\downarrow}B\}) = V \cup \{R{\downarrow}B,\ \bar{v}R{\downarrow}B\}$, it follows that $W = \Phi(V \cup \{R{\downarrow}B\}) = \Sigma_{conc}\Sigma_{subs}(V \cup \{R{\downarrow}B,\ \bar{v}R{\downarrow}B\})$.

■

REMARK In the proof of theorem 4.7 we essentially used the finiteness of all domains. If the domain is infinite, the $R{\downarrow}B$ of the "only if"-part does not need to exist. To the authors it seems that the "if"-part remains true even if #B is infinite. A proof of this case has not yet been found; we did not focus on it yet.

■

EXAMPLE 4.8 $L \underset{m}{\subsetneq} T_3 \underset{m}{\subsetneq} T_4$.

Let $A = \{a,b,c\}$ with $\#A = 3$. Let $B = \{a,b,c,d\}$ with $\#B = 4$.
Let $R{\downarrow}A := \bar{r}\{\langle a,b\rangle,\langle b,c\rangle,\langle c,a\rangle\}{\downarrow}A$ and
$R'{\downarrow}b = \bar{r}\{\langle a,b\rangle,\langle b,c\rangle,\langle c,d\rangle,\langle d,a\rangle,\langle a,c\rangle,\langle b,d\rangle\}{\downarrow}B$.
Note that (1) $R{\downarrow}A \in T_3 - L$, (2) $R'{\downarrow}B \in T_4 - T_3$, (3) for all $C \subset A$ with $C \neq \emptyset$, $(R{\downarrow}A)|C \in L$ and for all $D \subset B$ with $D \neq \emptyset$ $(R'{\downarrow}B)|D \in T_3$. Hence, from theorem 4.7 it follows that
$L \underset{m}{\subsetneq} \Sigma_{conc}\Sigma_{subs}(L \cup \{R{\downarrow}A,\ \bar{v}R{\downarrow}A\}) \subseteq T_3$ and
$T_3 \underset{m}{\subsetneq} \Sigma_{conc}\Sigma_{subs}(T_3 \cup \{R'{\downarrow}B,\ \bar{v}R'{\downarrow}B\}) \subseteq T_4$.

Now, since there is only one type of 3-cycle and one 4-cycle it follows that $\Sigma_{subs}(L \cup \{R{\downarrow}A,\ \bar{v}R{\downarrow}A\}) = \Sigma_{perm}(L \cup \{R{\downarrow}A\})$ and $\Sigma_{subs}(T_3 \cup \{\bar{v}R'{\downarrow}B,\ R'{\downarrow}B\}) = \Sigma_{perm}(T_3 \cup \{R'{\downarrow}B\})$. From the definition of T_k in example 3.9 it is clear that $T_3 \subseteq \Sigma_{conc}\Sigma_{perm}(L \cup \{R{\downarrow}A\})$ and $T_4 \subseteq \Sigma_{conc}\Sigma_{perm}(T_3 \cup \{R'{\downarrow}B\})$.
Hence $L \underset{m}{\subsetneq} \Sigma_{conc}\Sigma_{perm}(L \cup \{R{\downarrow}A\}) = T_3 \underset{m}{\subsetneq} \Sigma_{conc}\Sigma_{perm}(T_3 \cup (R'{\downarrow}B\})$
$= T_4$. ■

In the following section we develop more minimal extensions. As pointed out in section 1, when considering e.g. $L \underset{m}{\subsetneq} T_3$, the $R{\downarrow}A$ is not ordered linearly, but it is ordered according to the type of orderings in T_3. So, if we order according to the type of orderings in T_3, we admit 3-cycles in the orderings and we therefore admit new arrangements not yet present among the orderings of L. If we admit 4-cycles and consequently 3-cycles (See Moon [1968]), then we obtain all orderings of T_4. Moreover, if we admit all n-cycles, then we obtain the set of all antisymmetric and strongly complete relations which is T, the set of tournaments (See (3.7)). So, one might say that ordering boils down to arranging (according to the operations substitution, concatenation, restriction, conversion and permutation) several basic (dis)orders which cannot be analysed further (on the basis of these operations). The more (types of) basic (dis)orders one includes in this arrangement, the greater the set of orderings becomes.

§5 SEVERAL ORDERINGS

In this section several well-known as well as new sets of orderings will be classified. Along with this classification some basic results about orderings will be developed, employing some theorems of the foregoing sections.

First of all we show that whenever a set is classified as a set of orderings, then it contains a subset which is order isomorphic with $\mathbf{L}$, the set of linear orderings. So, ordering linearly is possible within any type of ordering; ordering linearly is the most restrictive way of ordering.

THEOREM 5.1 Let $V \subseteq \mathbf{R}$ be a classified set of orderings. Then there is a subset $W \subseteq V$ and an order-isomorphism $h : \mathbf{L} \to W$.

PROOF By corollary 2.4 there are two cases.

Case 1 All relations in V are reflexive.

Since V is classified as a set of orderings $\Sigma_{conc}(\{Id{\downarrow}\{x\} : x \in \mathbf{U}\}) \subseteq V$. It is elementary to prove that $\mathbf{L} = \Sigma_{conc}(\{Id{\downarrow}\{x\} : x \in \mathbf{U}\})$. Hence we are done by taking h equal to $\bar{I}$, and theorem 2.10.

Case 2 All relations in V are irreflexive.

Hence, all relations in $\overline{cv}(V)$ are reflexive. By case 1 it follows that $\mathbf{L} \subseteq \overline{cv}(V)$. So, by taking $W = \overline{cv}(\mathbf{L})$ and $h = \overline{cv}$, we are done by theorem 2.10. ∎

In theorem 5.1 we have shown that if V is a classified set of reflexive orderings, then $\mathbf{L} \subseteq V$; otherwise $\overline{cv}(\mathbf{L}) \subseteq V$. Therefore, $\mathbf{L}$ is the smallest (with respect to inclusion) classifiable set of orderings. Next we discuss all possible minimal extensions of $\mathbf{L}$. Using the order-isomorphisms $\bar{I}$, $\overline{cv}$ we then also have all minimal extensions of $\overline{cv}(\mathbf{L})$.

THEOREM 5.2 Let $V \subseteq R$ be classified as a set of orderings. Then $L \underset{m}{\subsetneq} V$ if and only if $V \in \{W, T_3, O_2\}$. Where $O_2 := \{R{\downarrow}A \in R : R{\downarrow}A$ is reflexive, antisymmetric,

$\langle\overline{nsc}^2, \overline{rs}\rangle$-transitive and $\langle\bar{a}^2, \bar{a}\rangle$-transitive$\}$.

PROOF Note that $O_2 = \Sigma_{conc}(\{\bar{r}\emptyset{\downarrow}X : X \in D \text{ and } \#X \in \{1,2\}\})$.

So, $O_2 = \Sigma_{conc}\Sigma_{subs}(L \cup \{\bar{r}\emptyset{\downarrow}Y, \overline{vr\emptyset{\downarrow}Y}\})$, where $Y \in D$ and $\#Y = 2$. The "if"-part now follows from theorem 4.7, example 4.2 and example 4.8.

("Only if"-part) Let $L \underset{m}{\subset} V$. Then there is a relation $R{\downarrow}B \in V - L$. We can distinguish three cases.

<u>Case 1</u> $R{\downarrow}B$ is not antisymmetric. Hence there are $x,y \in B$, $x \neq y$: $All{\downarrow}\{x,y\} = (R{\downarrow}B)|\{x,y\}$. Since V is a classified set of orderings $L \underset{m}{\subsetneq} \Phi(L \cup \{All{\downarrow}\{x,y\}\}) \subseteq V$. Hence $V = \Phi(L \cup \{All{\downarrow}\{x,y\}) = W$.

<u>Case 2</u> $R{\downarrow}B$ is not strongly complete. Since $L \subseteq V$ it follows that all relations in V are reflexive. So, there are $x,y \in B$, $x \neq y$: $\bar{r}\emptyset{\downarrow}\{x,y\} = (R{\downarrow}B)|\{x,y\}$. Similarly as in case 1 it follows that $L \underset{m}{\subsetneq} O_2 = \Phi(L \cup \{\bar{r}\emptyset{\downarrow}\{x,y\}\}) = V$.

<u>Case 3</u> $R{\downarrow}B$ is antisymmetric and strongly complete, but not transitive. So, $R{\downarrow}B \in T - L$. Hence there are $x,y,z \in B$, $\#\{x,y,z\} = 3$, such that $(R{\downarrow}B)|\{x,y,z\} = \bar{r}\{\langle x,y\rangle,\langle y,z\rangle,\langle z,x\rangle\}{\downarrow}\{x,y,z\}$. Similarly as in case 1 it follows that $L \underset{m}{\subsetneq} T_3 = \Phi(L \cup \{(R{\downarrow}B)|\{x,y,z\}\}) = V$.

■

EXAMPLE 5.3 Let for all $k \in \{1,2,\ldots\}$, $B_k \in D$ with $\#B_k = k$. For all $k \in \{2,3,4,\ldots\}$, $O_k := \Phi(L \cup \{\bar{r}\emptyset{\downarrow}B_k\})$. **(5.3.1)** Note that for all $D \subset B_{k+1}$ with $D \neq \emptyset$, $(\bar{r}\emptyset{\downarrow}B_{k+1})|D \in O_k$. So, $O_{k+1} = \Phi(L \cup \{\bar{r}\emptyset{\downarrow}B_{k+1}\}) \subseteq \Phi(O_k \cup \{\bar{r}\emptyset{\downarrow}B_{k+1}\}) \subseteq O_{k+1}$. Hence, by theorem 4.7, $L \underset{m}{\subsetneq} O_2 \underset{m}{\subsetneq} O_3 \ldots O_k \underset{m}{\subsetneq} O_{k+1} \ldots \subset O_\infty$ **(5.3.2)** where $O_\infty := \cup\{O_k : k \in \{2,3,\ldots\}\}$. Note that O_∞ is the set of reflexive, antisymmetric and $\langle\overline{cav}^2, \overline{cav}\rangle$-transitive relations.

Hence, O_∞ can be classified as a set of orderings.

■

Next we will study sets of strongly complete relations. To be precise we will study subsets of **Q** and **T**. When doing so it becomes immediately clear that it is even impossible to indicate all subsets of **Q** or **T** which can be classified as sets of orderings. We start off with the strongly complete and antisymmetric relations.

DEFINITION 5.4 <u>IRREDUCIBLE</u> (See Moon [1968])

$R{\downarrow}A \in \mathbf{R}$ is said to be <u>irreducible</u> if for every non-trivial subset B of A (i.e. $\emptyset \neq B \subset A$), $R{\downarrow}A \neq (R{\downarrow}A)|B \gg (R{\downarrow}A)|(A-B)$.

■

It is a well-known fact (see e.g. Moon [1968]) that for $R{\downarrow}A \in \mathbf{T}$, $R{\downarrow}A$ is irreducible if and only if there is a Hamilton-cycle along $R{\downarrow}A$, i.e. if and only if $All{\downarrow}A = [R{\downarrow}A]^{\#A}$. For any $R{\downarrow}A \in \mathbf{T}$, with $\#A$ finite, we can find a partition $A_1, A_2, ..., A_k$ of A such that $(R{\downarrow}A)|A_i$ is irreducible for all $i \in \{1,2,..,k\}$ and $R{\downarrow}A = (R{\downarrow}A)|A_1 \gg (R{\downarrow}A)|A_2 \gg ... \gg (R{\downarrow}A)|A_k$. Now let $V \subseteq \mathbf{T}$ be a classified set of orderings. Let $I(V) := \{R{\downarrow}A \in V : R{\downarrow}A \text{ is irreducible}\}$. Then we obviously have $V = \Sigma_{conc}(I(V))$. Moreover, we have the following theorem.

THEOREM 5.5 Let $V \subseteq I(\mathbf{T})$ be non-empty, such that V is closed under conversion and substitution and for all $R{\downarrow}A \in V$ and for all $B \subset A$ with $B \neq \emptyset$, if $(R{\downarrow}A)|B$ is irreducible, then $(R{\downarrow}A)|B \in V$. Let $W \subseteq \mathbf{T}$. $W = \Sigma_{conc}(V)$ if and only if W can be classified as a set of orderings and $I(W) = V$.

PROOF Since $W = \Sigma_{conc}(I(W)) = \Sigma_{conc}(V)$ for classifiable sets $W \subseteq \mathbf{T}$, the "if"-part is evident.

("Only if"-part) Suppose $W = \Sigma_{conc}(V)$. Then $V \subseteq W$. So, $V \subseteq I(W)$. Obviously, $I(W) \subseteq V$. Therefore $V = I(W)$. Since V is a subset of **T** and is closed under conversion and substitution, it follows by lemma 4.4 and lemma 4.5 that

$\Sigma_{conv}\Sigma_{conc}(V) \subseteq \Sigma_{conc}\Sigma_{conv}(V) \subseteq \Sigma_{conc}(V)$ and

$\Sigma_{subs}\Sigma_{conc}(V) \subseteq \Sigma_{conc}\Sigma_{subs}(V) \subseteq \Sigma_{conc}(V)$. Since for all $w \in \{conc, conv, rest, subs, perm\}$ and for all $X \subseteq \mathbf{R}$: $X \subseteq \Sigma_w(X)$ it follows that $\Sigma_{conv}\Sigma_{conc}(V) = \Sigma_{conc}(V)$ and $\Sigma_{subs}\Sigma_{conc}(V) = \Sigma_{conc}(V)$. Hence, by lemma 4.4 it follows that $\Sigma_{conc}(V)$ is closed under substitution and conversion. Furthermore, $\Sigma_{conc}(V)$ is closed under concatenation. Now by the assumptions on V it follows that $\mathbf{N}(W) = \mathbf{N}(V) = \{Id{\downarrow}\{x\} : x \in \mathbf{U}\}$. So, by theorem 2.3 we are done if W is closed under restriction. Let $A, B \in \mathbf{D}$ with $\emptyset \neq B \subseteq A$. Let $R{\downarrow}A \in W$. It is sufficient to prove that $(R{\downarrow}A)|B \in W$. Since $R{\downarrow}A \in W = \Sigma_{conv}(V)$ there are $R^1{\downarrow}A^1$, $R^2{\downarrow}A^2$, ... $R^k{\downarrow}A^k \in V$ such that for all $1 \leq i < j \leq k$ $A^i \cap A^j = \emptyset$ and $R{\downarrow}A = R^1{\downarrow}A^1 \gg R^2{\downarrow}A^2 \gg \ldots \gg R^k{\downarrow}A^k$. Now there are $i_1, i_2, \ldots, i_l \in \{1, 2, \ldots, k\}$ such that for all $j \in \{1, 2, \ldots, k\}$: $j \in \{i_1, i_2, \ldots, i_l\}$ iff $B^j := A^j \cap B \neq \emptyset$. So, $(R{\downarrow}A)|B = (R^{i_1}{\downarrow}A^{i_1})|B^{i_1} \gg (R^{i_2}{\downarrow}A^{i_2})|B^{i_2} \gg \ldots \gg (R^{i_l}{\downarrow}A^{i_l})|B^{i_l}$. Since by the assumptions on V, $(R^j{\downarrow}A^j)|B^j \in \Sigma_{conc}(V)$ for all $j \in \{i_1, i_2, \ldots, i_l\}$ it follows that $(R{\downarrow}A)|B \in \Sigma_{conc}(V) = W$.

■

EXAMPLE 5.6 <u>TOURNAMENTS</u>

Let $V_n := \{R{\downarrow}A \in I(\mathbf{T}) : \#A \leq n\}$. Note that V_n is non-empty, closed under conversion and substitution and for all $R{\downarrow}A \in V_n$ and for all $B \subseteq A$ with $B \neq \emptyset$, if $(R{\downarrow}A)|B$ is irreducible then $(R{\downarrow}A)|B \in V_n$. So, by theorem 5.5, $\Sigma_{conc}(V_n)$ can be classified as a set of orderings. Furthermore, $I(\mathbf{T}_n) = V_n$.

Hence, $\mathbf{T}_n = \Sigma_{conc}(V_n)$. (See also example 3.9). **(5.6.1)**

We already know that $\mathbf{L} \underset{m}{\subsetneq} \mathbf{T}_3 \underset{m}{\subsetneq} \mathbf{T}_4$ (example 4.8). A natural question to raise is whether there is a minimal extension of $\mathbf{T}_4$. Or, more generally, is there a minimal extension of $\mathbf{T}_n$ for each n?

Consider $R^n{\downarrow}A^n$ for $n \in \{3, 4, 5, \ldots\}$ such that $A_n = \{a_1, a_2, \ldots a_n\}$ and $\langle a_i, a_j\rangle \in R^n{\downarrow}A^n$ iff $j \geq i$ and $\langle 1, n\rangle \neq \langle i, j\rangle$ or $\langle i, j\rangle = \langle n, 1\rangle$. Let $W_n := \Sigma_{perm}\Sigma_{conv}(\{R^n{\downarrow}A^n\})$. Now consider

$V_n \cup W_{n+1}$; $V_n \cup W_{n+1}$ is closed under conversion and substitution and for all $R{\downarrow}A \in V_n \cup W_{n+1}$ and for all $B \subseteq A$ with $B \neq \emptyset$, if $(R{\downarrow}A)|B$ is irreducible, then $(R{\downarrow}A)|B \in V_n \cup W_{n+1}$. Hence, by theorem 5.5, $T_{n+1,1} := \Sigma_{conc}(V_n \cup W_{n+1})$ can be classified as a set of orderings. **(5.6.2)**

Now $R^{n+1}{\downarrow}A^{n+1} \notin T_n$.

So $T_n = \Sigma_{conc}(V_n) \subset \Sigma_{conc}(V_n \cup W_{n+1}) = T_{n+1,1}$.

Since $V_n \cup W_{n+1} \subseteq T_n \cup W_{n+1} \subseteq \Sigma_{perm}\Sigma_{conv}(T_n \cup \{R^{n+1}{\downarrow}A^{n+1}\})$

$$\subseteq \Phi(T_n \cup \{R^{n+1}{\downarrow}A^{n+1}\}),$$

it follows that $T_n \underset{m}{\subsetneq} T_{n+1,1}$ by theorem 4.7.

Since $W_n \subseteq V_n$ we have that $T_{n,1} \subseteq T_n$. There is only one type of irreducible tournament on domains with either 1,3 or 4 elements. Therefore $W_1 = V_1$, $W_3 = V_3$ and $W_4 = V_4$. Furthermore, $W_n \subset V_n$, if $n \geq 5$ by which it follows that $T_{n,1} \subset T_n$ for $n \geq 5$. So, T_n is not a minimal extension of T_{n-1} if $n \geq 5$.

■

It is possible to develop more classifiable sets of tournaments (see Storcken [1989]), but because of time and space limitations the investigations on classifiable sets of tournaments is stopped here.

So far we have developed the outer parts of the inclusion diagram at the end of this section. Next we will develop classifiable subsets which are strongly complete and quasi-transitive. So, classifiable subsets of the set of quasi-orderings.

First we determine classifiable subsets of the set of semi-orderings. Next we determine classifiable subsets of the set of interval orderings.

EXAMPLE 5.7 <u>SEMI-ORDERINGS</u>

Let $\mathbf{S} := \{R{\downarrow}A \in \mathbf{C} : \langle\bar{a}\ \overline{rs}\ \bar{a},\ \bar{a}\rangle$-transitive and

$\langle\bar{a}\ \bar{a}\ \overline{rs},\ \bar{a}\rangle$-transitive$\}$. **(5.7.1)**

$\mathbf{S}$ is called the <u>set of semi-orderings</u>. It is classifiable by theorem 3.5 and 2.11. We will now determine classifiable sets of relations, between $\mathbf{W}$ and $\mathbf{S}$. For $k \in \{1,2,\dots\}$, consider $A^k := \{a_1,a_2,\dots,a_k\}$ in D and $\tilde{R}^k{\downarrow}A^k$ such that $\langle a_i,a_j\rangle \in \tilde{R}^k{\downarrow}A^k$ iff $i \le j+1$ for all $i,j \in \{1,2,\dots,k\}$.

Let $\mathbf{S}_k := \{R{\downarrow}A \in \mathbf{S} : \langle\overline{rs}^k,\overline{rs}^{k-1}\rangle$-transitive$\}$ and **(5.7.2)**

$\tilde{\mathbf{S}}_k := \Phi(\mathbf{S}_k \cup \{\tilde{R}^{k+1}{\downarrow}A^{k+1}\})$ for all $k \in \{2,3,\dots\}$ **(5.7.3)**

Note that $\mathbf{W} = \mathbf{S}_2$, that $\mathbf{S}_k$ and $\tilde{\mathbf{S}}_k$ can be classified as sets of orderings and that $\mathbf{S}_k \underset{m}{\subsetneq} \tilde{\mathbf{S}}_k$ for all $k \in \{2,3,4,\dots\}$.

Furthermore, let $\bar{R}^k{\downarrow}(A^k \cup \{y\})$ consist of those pairs $\langle a,b\rangle$ which are either in $\tilde{R}^k{\downarrow}A^k$ or for which $\{a,b\} = \{a_1,y\}$ or $a = y$. It is straightforward to prove that $\bar{R}^k{\downarrow}(A_k \cup \{y\}) \in \mathbf{S}_{k+1} - \tilde{\mathbf{S}}_k$.

Hence, $\mathbf{W} = \mathbf{S}_2 \underset{m}{\subsetneq} \tilde{\mathbf{S}}_2 \subset \mathbf{S}_3 \underset{m}{\subsetneq} \tilde{\mathbf{S}}_3 \subset \dots\ \mathbf{S}_k \underset{m}{\subsetneq} \tilde{\mathbf{S}}_k \subset \mathbf{S}_{k+1} \subset \dots \subset \mathbf{S}$.

Now $\cup\{\mathbf{S}_k : k \in \{2,3,\dots\}\} = \mathbf{S}$ since for all $R{\downarrow}A \in \mathbf{S}$ there is a $k \ge \#A$ such that $R{\downarrow}A$ is $\langle\overline{rs}^k,\overline{rs}^{k-1}\rangle$-transitive.

For $k \in \{2,3,4,\dots\}$. Let $\bar{\bar{\mathbf{S}}}_k := \Phi(\mathbf{W} \cup \{\tilde{R}^k{\downarrow}A^k\})$ and **(5.7.4)**

$\bar{\mathbf{S}}_k := \{R{\downarrow}A \in \mathbf{C} : R{\downarrow}A$ is $\langle\bar{a}\ \overline{rs}\ \bar{a}\ \overline{rs},\ \bar{a}\rangle$-transitive and $\langle\overline{rs}^k,\overline{rs}^{k-1}\rangle$-transitive$\}$. Then $\bar{\bar{\mathbf{S}}}_k \subseteq \bar{\mathbf{S}}_k \subset \mathbf{S}_{k+1}$ for all $k \in \{2,3,4,\dots\}$.

On the other hand, it is elementary, although cumbersome, to prove that $\bar{\mathbf{S}}_k \subseteq \bar{\bar{\mathbf{S}}}_k$ for all $k \in \{2,3,\dots\}$. Hence, $\bar{\mathbf{S}}_k = \bar{\bar{\mathbf{S}}}_{k+1}$ for all $k \in \{2,3,\dots\}$. Note that $(\tilde{R}^k{\downarrow}A^k)|B \in \bar{\bar{\mathbf{S}}}_{k-1}$ for all $k \in \{3,4,5\dots\}$ and for all $B \subseteq A^k$ with $B \ne \emptyset$. Hence, by theorem 4.7, $\bar{\mathbf{S}}_k \underset{m}{\subsetneq} \Phi(\bar{\mathbf{S}}_k \cup \{\tilde{R}^{k+1}{\downarrow}A^{k+1}\})$ for all $k \in \{2,3,4\dots\}$.

Since $\mathbf{W} \subseteq \bar{\bar{\mathbf{S}}}_k$, we have for all $k \in \{3,4,5,\dots\}$

$\bar{\mathbf{S}}_{k-1} \subset \bar{\bar{\mathbf{S}}}_k = \Phi(\mathbf{W} \cup \{\tilde{R}^k{\downarrow}A^k\}) \subseteq \Phi(\bar{\mathbf{S}}_{k-1} \cup \{\tilde{R}^k{\downarrow}A^k\})$. So,

$\bar{\mathbf{S}}_{k+1} = \Phi(\bar{\mathbf{S}}_k \cup \{R^{k+1}{\downarrow}A^{k+1}\})$ for all $k \in \{2,3,4,\dots\}$ and

$\mathbf{W} \underset{m}{\subsetneq} \bar{\mathbf{S}}_2 = \bar{\mathbf{S}}_3 \underset{m}{\subsetneq} \bar{\mathbf{S}}_4 \underset{m}{\subsetneq} \bar{\mathbf{S}}_5 \subset \dots \subset \bar{\mathbf{S}}_\infty :=$

$\cup\{\bar{\mathbf{S}}_k : k \in \{2,3,4,\dots\}\}$. **(5.7.5)**

Note that $\bar{\mathbf{S}}_\infty = \{R{\downarrow}A \in \mathbf{R} : R{\downarrow}A$ is strongly complete and

$\langle\bar{a}\ \overline{rs}\ \bar{a}\ \overline{rs},\bar{a}\rangle$-transitive$\}$. ∎

EXAMPLE 5.8 INTERVAL ORDERINGS

(5.8.1)

Let $\mathbf{I} := \{R{\downarrow}A \in \mathbf{C} : R{\downarrow}A$ is $\langle\bar{a}\ \overline{rs}\ \bar{a},\bar{a}\rangle$-transitive$\}$.

$\mathbf{I}$ is called the set of interval orderings. By 2.11 and 3.5 it can be classified as a set of orderings. Note that $\mathbf{L} \underset{m}{\subsetneq} \mathbf{W} \subset \mathbf{S}_3 \subset \ldots \subset \mathbf{S} \subset \mathbf{I} \subset \mathbf{Q}$. In example 5.7 we classified sets of orderings which were between $\mathbf{W}$ and $\mathbf{S}$. In this example we classify sets of orderings between $\mathbf{S}$ and $\mathbf{I}$. (See the diagram at the end of this section). For all positive integers k let

$\mathbf{I}_k := \{R{\downarrow}A \in \mathbf{I} : R{\downarrow}A$ is $\langle\bar{a}^k\ \overline{rs},\bar{a}\rangle$-transitive$\}$. (5.8.2)

Since $\langle\bar{a}^k\ \overline{rs},\bar{a}\rangle$-transitivity implies $\langle\bar{a}^l\ \overline{rs},\bar{a}\rangle$-transitivity and not conversely whenever $k < l$, it follows that $\mathbf{I}_1 \subset \mathbf{I}_2 \subset \mathbf{I}_3 \subset \ldots \subset \mathbf{I}$. Furthermore, it is easy to prove that $\mathbf{I}_1 = \mathbf{W}$ and $\mathbf{I}_2 = \mathbf{S}$.

Note that $\mathbf{I} = \cup\{\mathbf{I}_k : k \in \{1,2,3,\ldots\}\}$.

Now consider $\hat{R}^k{\downarrow}B^k \in \mathbf{R}$ for $k \in \{2,3,4,\ldots\}$, where $B^k = \{b_0,b_1,b_2,\ldots,b_k\} \in \mathbf{D}$ and $\langle b_i,b_j\rangle \in \hat{R}^k{\downarrow}B^k$ iff $i \le j$ or $j = 0$. Clearly, we have $\hat{R}^k{\downarrow}B^k \in \mathbf{I}_k - \mathbf{I}_{k-1}$ for all $k \in \{2,3,\ldots\}$ and $(\hat{R}^k{\downarrow}B^k)|C \in \mathbf{I}_{k-1}$ for all $C \subset B^k$ with $C \neq \emptyset$.

Hence, by theorem 4.7, $\mathbf{I}_{k-1} \underset{m}{\subsetneq} \tilde{\mathbf{I}}_k := \Phi(\mathbf{I}_{k-1} \cup \{\hat{R}^k{\downarrow}B^k\})$ for all $k \in \{2,3,\ldots\}$. (5.8.3)

Moreover, it is straightforward to prove that $\tilde{\mathbf{I}}_k \subset \mathbf{I}_k$ for all $k \in \{2,3,4,\ldots\}$. Hence, we have

$\mathbf{W} = \mathbf{I}_1 = \mathbf{S}_2 \underset{m}{\subsetneq} \tilde{\mathbf{S}}_2 = \tilde{\mathbf{I}}_2 \subset \mathbf{I}_2 = \mathbf{S} \underset{m}{\subsetneq} \tilde{\mathbf{I}}_3 \subset \mathbf{I}_3 \underset{m}{\subsetneq} \tilde{\mathbf{I}}_4 \ \ldots \subset \mathbf{I}$.

Now consider $\hat{\mathbf{I}}_k := \Phi(\mathbf{W} \cup \{\hat{R}^k{\downarrow}B^k\})$ for $k \in \{2,3,4,\ldots\}$. (5.8.4)

Then it is easy to prove that

$\mathbf{W} \underset{m}{\subsetneq} \hat{\mathbf{I}}_2 = \tilde{\mathbf{S}}_2 \underset{m}{\subsetneq} \hat{\mathbf{I}}_3 \underset{m}{\subsetneq} \hat{\mathbf{I}}_4 \underset{m}{\subsetneq} \ldots \subset \hat{\mathbf{I}}_\infty$, where

$\hat{\mathbf{I}}_\infty := \cup\{\hat{\mathbf{I}}_k : k \in \{2,3,\ldots\}\}$. Furthermore, although (5.8.5) the proof is cumbersome and therefore omitted here, it is straightforward to prove that

$\hat{I}_\infty = \{R{\downarrow}A \in I$: $R{\downarrow}A$ is $<\overline{rs}^3,\overline{rs}^2>$-transitive$\}$. So, $\hat{I}_\infty$ is classified as a set of orderings.

■

We have studied several classifiable sets of orderings between **W** and **S** and between **S** and **I**. Next we will study classifiable sets of orderings between **I** and **Q**. We will summarize all sets of classifiable orderings discussed up till now in an inclusion diagram at the end of this section.

EXAMPLE 5.9 QUASI - ORDERINGS

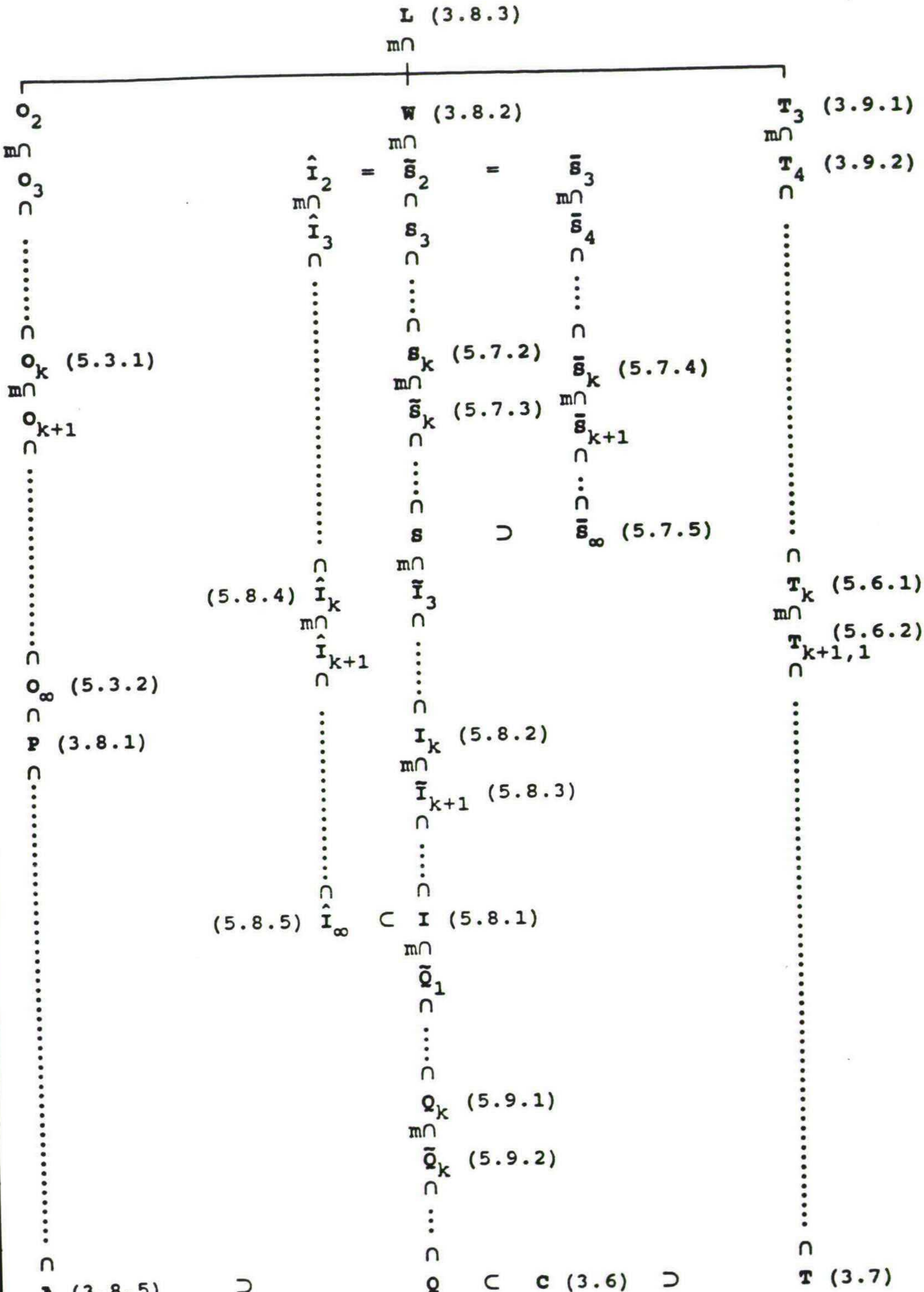
For $k \in \{1,2,3,..\}$, let $Q_k := \{R{\downarrow}A \in Q$: $R{\downarrow}A$ is $<\bar{a}^k\ \overline{rs}\ \bar{a}^k,\bar{a}>$-transitive$\}$. **(5.9.1)**

Then Q_k is classifiable as a set of orderings for all $k \in \{1,2,3,...\}$. Since $<\bar{a}^k\ \overline{rs}\ \bar{a}^k,\bar{a}>$-transitivity implies $<\bar{a}^l\overline{rs}\bar{a}^l,\bar{a}>$-transitivity and not conversely if $k < l$, it follows that $Q_1 = I \subset Q_2 \subset Q_3 \ldots \subset Q$. Note that $Q = \cup\{Q_k : k \in \{1,2,3,..\}\}$.

For all $k \in \{1,2,...\}$ let $C^k := \{a_1,a_2,...,a_{2k}\}$ in **D** and $R^k{\downarrow}C^k$ be defined as follows. For all $i,j \in \{1,2,...,2k\}$: $<a_i,a_j> \in R^k{\downarrow}C^k$ iff $i \le j$ or $i+j$ is odd. Then it follows again that $Q_1 = I \underset{m}{\subsetneq} \tilde{Q}_1 \subset Q_2 \underset{m}{\subsetneq} \tilde{Q}_2 \subset \ldots \subset Q_k \underset{m}{\subsetneq} \tilde{Q}_k \subset \ldots \subset Q$, **(5.9.2)**

where $\tilde{Q}_k := \Phi(Q_k \cup \{R^{k+1}{\downarrow}C^{k+1}\})$.

■


L (3.8.3)
W (3.8.2)
T3 (3.9.1)
T4 (3.9.2)
O2
O3
Ok (5.3.1)
Ok+1
O∞ (5.3.2)
P (3.8.1)
A (3.8.5)
Î2 = S̃2 = S̄3
Î3
S3
S̄4
Sk (5.7.2)
S̃k (5.7.3)
S̄k (5.7.4)
S̄k+1
S ⊃ S̄∞ (5.7.5)
(5.8.4) Îk
Îk+1
Ĩ3
Ik (5.8.2)
Ĩk+1 (5.8.3)
(5.8.5) Î∞ ⊂ I (5.8.1)
Q̃1
Qk (5.9.1)
Q̃k (5.9.2)
Tk (5.6.1)
Tk+1,1 (5.6.2)
Q ⊂ C (3.6) ⊃ T (3.7)

§ 6 DISCUSSION

In the preceding sections we have introduced a theory of orderings, i.e. we have defined under what conditions a set of relations should be called a set of orderings. No similar or related theory is known to the authors. In section 2 we tried to explain our motivation for choosing the specific criteria presented before. Here we try to answer the question whether our theory of orderings is satisfactory. We will show that the six criteria for a set of orderings are independent and that certain variations of the criteria lead to undesired results.

First of all, it has been pointed out in the preceding sections that all well-known sets of orderings can be classified as such, in other words that they satisfy the six criteria mentioned. Furthermore, it has been shown in section 4 that $\mathbf{L} \subset_{m} \mathbf{W}$. So, the criteria of section 2 exclude the existence of a classifiable set of orderings between $\mathbf{L}$ and $\mathbf{W}$. Hence, the first two conditions posed in section 1 are satisfied by the six criteria. But in section 5 it becomes clear that these criteria are satisfied by many other sets of relations and one might raise the question if we accept intuitively all these sets of relations as sets of orderings. Stated otherwise, the six criteria of section 2 could appear to be too weak in order to model orderings. The authors, however, could not find any "reasonable" extra conditions, which would decrease the number of classifiable sets of orderings. Moreover, the existence of only 8 different ordermorphisms (corollary 2.9) makes clear that these six criteria are at least at first sight not too weak.

That the six criteria are independent of each other is shown in the following six examples.

EXAMPLE 6.1 <u>TRIVIALITY</u>

$\mathbf{R}$ itself does not satisfy non-triviality, but it is closed under permutation, conversion, restriction, concatenation and substitution. ■

EXAMPLE 6.2 <u>NOT CLOSED UNDER PERMUTATION</u>

Let $x \in \mathbf{U}$. Consider $W := \{R{\downarrow}A \in \mathbf{R} :$ There is a relation $R'{\downarrow}A$ in $\mathbf{L}$ such that $R{\downarrow}A = R'{\downarrow}A - Id{\downarrow}\{x\}\}$. Note that $R'{\downarrow}A = R{\downarrow}A$ if $x \notin A$. Clearly W is not trivial nor closed under permutation, but it is closed under conversion, restriction, concatenation and substitution.

■

EXAMPLE 6.3 <u>NOT CLOSED UNDER CONVERSION</u>

Let $W := \{R{\downarrow}A \in \mathbf{R} : R{\downarrow}A$ is reflexive and for all $x,y,z \in A$:

If $<x,y> \in \bar{a}R{\downarrow}A$ and $<y,z> \overline{\in}\, rsR{\downarrow}A$, then $<x,z> \in \bar{a}R{\downarrow}A\}$. Let $R{\downarrow}X := \bar{r}\{<b,a>, <b,c>, <c,b>\}{\downarrow}X$, where $X = \{a,b,c\}$ and $\#X = 3$. Then $R{\downarrow}X \in W$ and $\bar{v}R{\downarrow}X \notin W$. So, W is not closed under conversion, but it is straightforward to prove that W satisfies the other criteria.

■

EXAMPLE 6.4 <u>NOT CLOSED UNDER RESTRICTION</u>

Let $W := \{R{\downarrow}A \in \mathbf{W} :$ For all $B \subseteq A$, $B \neq \varnothing$: If $(R{\downarrow}A)|B$, then $\#B \neq 2\}$. Clearly W is not closed under restriction, but it satisfies the other criteria.

■

EXAMPLE 6.5 <u>NOT CLOSED UNDER CONCATENATION</u>

Let $W := \{All{\downarrow}X \in \mathbf{R} : X \in \mathbf{D}\}$. W satisfies criterion 2.2.1, 2.2.2, 2.2.3, 2.2.4 and 2.2.6. W is obviously not closed under concatenation.

■

EXAMPLE 6.6 NOT CLOSED UNDER SUBSTITUTION

Let $W := \{R{\downarrow}A \in \mathbf{R} :$ For all $B \subseteq A$ with $B \neq \emptyset$: if $(R{\downarrow}A)|B = All{\downarrow}B$, then $\#B \leq 2\}$. W is not closed under substitution, but it does satisfy criterion 2.2.1 up to 2.2.5.

■

In section 2 we formulated our intuitive motivation for the six criteria. Next, we will discuss variations of the five closedness conditions, since intuition might be modelled by different conditions. The examples 6.1 up to 6.6 show that dropping any one of the six criteria would result in "odd" classifiable sets of orderings. Since the authors want rather less than more sets to be classifiable as sets of orderings, in what follows we will only discuss strengthenings of the criteria. In modelling the irrelevance of names of the ordered elements, we were not able to formulate other reasonable criteria than the closedness under permutation. This holds similarly for the closedness under conversion and restriction. Therefore, we only discuss variations of the closedness under concatenation and substitution. Furthermore, we remark that the discussed variations are not yet presented in a systematic way; other relevant varieties might exist but not yet recognized by us.

Let $A \in \mathbf{D}$ and let $R{\downarrow}A \in \mathbf{R}$. For $x \in A$ let $better(x,R{\downarrow}A) := \{y \in A : \langle y,x\rangle \in \bar{a}R{\downarrow}A\}$ be <u>the set of all elements</u> y <u>better</u> (with respect to $R{\downarrow}A$) <u>than</u> x. The set of <u>maximal elements of</u> $R{\downarrow}A$ are those for which there is no better element: $Max(R{\downarrow}A) := \{x \in A : better(x,R{\downarrow}A) = \emptyset\}$. The <u>minimal elements of</u> $R{\downarrow}A$ are the maximal elements of $\bar{v}R{\downarrow}A$: $Min(R{\downarrow}A) := Max(\bar{v}R{\downarrow}A)$. The <u>top elements of</u> $R{\downarrow}A$ are those maximal elements of $R{\downarrow}A$ which are better than at least one other element: $Top(R{\downarrow}A) := \{x \in A : x \in Max(R{\downarrow}A)$ and there is a $y \in A$ such that $x \in better(y,R{\downarrow}A)\}$. The <u>bottom elements of</u> $R{\downarrow}A$ are the top elements of $\bar{v}R{\downarrow}A$: $Bottom(R{\downarrow}A) = Top(\bar{v}R{\downarrow}A)$.

On the basis of these four notions one might define four new concatenation operations.

DEFINITION 6.7 <u>CONCATENATION OPERATIONS</u>

Let $R{\downarrow}A,\ R'{\downarrow}B \in \mathbf{R}$.

If $Max(R'{\downarrow}B) \cap Min(R{\downarrow}A) = C = A \cap B$, $C \neq \emptyset$ and $(R{\downarrow}A)|C = (R'{\downarrow}B)|C$, then

$R{\downarrow}A \gg_1 R'{\downarrow}B := \{\langle x,y\rangle \in (A \cup B) \times (A \cup B) : \langle x,y\rangle \in R{\downarrow}A$ or $\langle x,y\rangle \in R'{\downarrow}B$ or $\langle x,y\rangle \in (A - C) \times (B - C)\}{\downarrow}(A \cup B)$.

If $Max(R'{\downarrow}B) = Min(R{\downarrow}A) = C = A \cap B$, $C \neq \emptyset$ and $(R{\downarrow}A)|C = (R'{\downarrow}B)|C$, then $R{\downarrow}A \gg_2 R'{\downarrow}B := R{\downarrow}A \gg_1 R'{\downarrow}B$.

If $Top(R'{\downarrow}B) \cap Bottom(R{\downarrow}A) = C = A \cap B$, $C \neq \emptyset$ and $(R{\downarrow}A)|C = (R'{\downarrow}B)|C$, then $R{\downarrow}A \gg_3 R'{\downarrow}B :=$ $\{\langle x,y\rangle \in (A \cup B) \times (A \cup B) : \langle x,y\rangle \in R{\downarrow}A$ or $\langle x,y\rangle \in R'{\downarrow}B$ or $\langle x,y\rangle \in (A - C) \times (B - C)\}{\downarrow}(A \cup B)$.

If $Top(R'{\downarrow}B) = Bottom(R{\downarrow}A) = C = A \cap B$, $C \neq \emptyset$ and $(R{\downarrow}A)|C = (R'{\downarrow}B)|C$, then $R{\downarrow}A \gg_4 R'{\downarrow}B := R{\downarrow}A \gg_3 R'{\downarrow}B$.

EXAMPLE 6.8

Let $A = \{a,b,c,d\}$ and $B = \{c,d,e,f\}$, where $\#A = \#B = 4$. Let

$R{\downarrow}A = \overline{cav}\{\langle a,d\rangle,\langle a,c\rangle,\langle b,c\rangle\}{\downarrow}A$ and

$R'{\downarrow}B := \overline{cav}\{\langle c,e\rangle,\langle c,f\rangle,\langle d,f\rangle\}{\downarrow}B$. Note that $R{\downarrow}A,\ R'{\downarrow}B \in \mathbf{S}$. Furthermore, note that $Max(R'{\downarrow}B) = Top(R'{\downarrow}B) = \{c,d\} = Min(R{\downarrow}A) = Bottom(R{\downarrow}A)$. So $R{\downarrow}A \gg_i R'{\downarrow}B = R{\downarrow}A \gg_j R'{\downarrow}B$ for all

$i,j \in \{1,2,3,4\}$. Now $(R{\downarrow}A) \gg_1 R'{\downarrow}B)|\{b,c,d,e\}$ is not $\langle \bar{a}^2\overline{rs},\bar{a}\rangle$-transitive. Therefore, $R{\downarrow}A \gg_i R'{\downarrow}B \notin \mathbf{S}$ for all $i \in \{1,2,3,4\}$.

■

The previous example shows that $\mathbf{S}$, being a well-known set of orderings, is not closed under $\gg_i$ for all $i \in \{1,2,3,4\}$. Therefore, these concatenation operations are not suitable in order to classify sets of orderings. In Storcken [1989] three other concatenation operations are discussed. Again all three appear not to be suitable in a classification system for orderings.

In order to discuss variations of the substitution operator we introduce two new substitution operators. For $R'{\downarrow}X$, $R{\downarrow}Y \in \mathbf{R}$, with $X \cap Y = \emptyset$ and $x \in X$, let $Sub_1(R'{\downarrow}X,x,R{\downarrow}Y) :=$ $Sub(R'{\downarrow}X,x,R{\downarrow}Y)$. And for $R'{\downarrow}X$, $R{\downarrow}Y \in \mathbf{R}$, with $\bar{v}R{\downarrow}Y = R{\downarrow}Y$ and $X \cap Y = \emptyset$, and for $x \in X$, let $Sub_2(R'{\downarrow}X,x,R{\downarrow}Y) =$ $Sub(R'{\downarrow}X,x,R{\downarrow}Y)$. Note that if $V \subseteq \mathbf{R}$ is closed under Sub_1 or Sub_2, then V satisfies criterion 2.2.6.

EXAMPLE 6.9

Let $X = \{x,y\}$, $Y = \{a,b\}$, $X \cap Y = \emptyset$ and $\#X = \#Y = 2$. Consider $All{\downarrow}X$ and $\bar{r}\emptyset{\downarrow}Y$, two reflexive and transitive relations. So, $All{\downarrow}X$, $\bar{r}\emptyset{\downarrow}Y \in \mathbf{P}$, the set of partial orderings.
But $Sub_1(All{\downarrow}X,x,r\emptyset{\downarrow}Y) = Sub_2(All{\downarrow}X,x,r\emptyset{\downarrow}Y)$ is not transitive. ■

Example 6.9 shows that $\mathbf{P}$, being a well-known set of orderings, is neither closed under Sub_1 nor under Sub_2. Therefore, Sub_1 and Sub_2 are not suitable variations of the substitution operator.

SUMMARY

We have formulated and motivated six criteria which a set of relations has to satisfy in order to be classified as a set of orderings. We have seen that all well-known sets of orderings can be classified as such and that a number of variations in the formulation of the criteria are inappropriate in the sense that some well-known sets of orderings would not be classifiable in that case. In section 3 we have generalized the transitivity condition and shown that this generalization suffices in order to classify a set of relations as a set of orderings. In section 4 we have characterized the minimal extensions of a given classified set of orderings. And in section 5 we developed an inclusion schema of many sets of orderings.

A major drawback of our theory may be that there are so many classifiable sets of orderings. It would be nice to find other plausible criteria, such that on the one hand all well-known sets of orderings satisfy these criteria and on the other hand there would be less classifiable sets of orderings.

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